$ax^2+bx+c=0$

§4.1 Polynomial Functions and Models

A polynomial function is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers and n is a nonnegative integer.

Determine which of the following are polynomial functions. For those that are, state the degree; for those that are not, tell why not. $\sqrt{3}\,x^2$

(a) $f(x) = 2 - 3x^4$

yes $-3x^4+2$

(b) $g(x) = \sqrt{x} = x^{\frac{1}{2}}$

no

(c) $h(x) = \dfrac{x^2 - 2}{x^3 - 1}$

no division with variables

(d) $F(x) = 0$

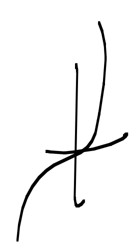
(e) $G(x) = 8$

(f) $H(x) = -2x^3(x - 1)^2$

$-2x^3(x^2-2x+1)$

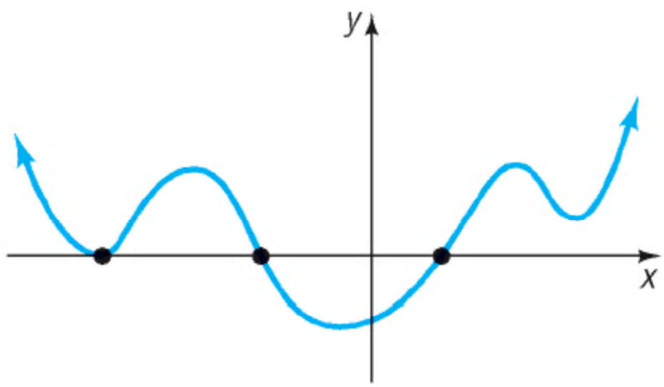
$= -2x^5+4x^4-2x^3$

x^3

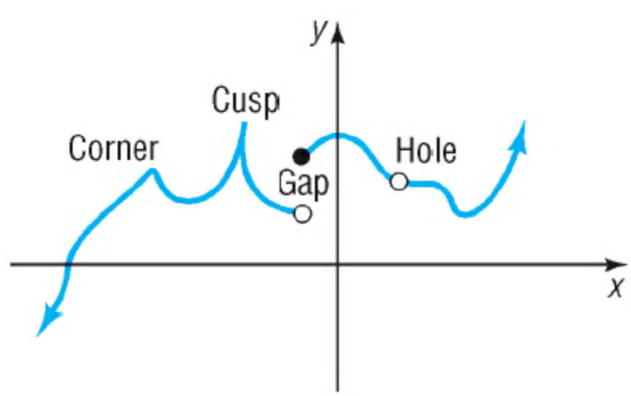


Summary of the Properties of the Graphs of Polynomial Functions

Degree	Form	Name	Graph
No degree	$f(x) = 0$	Zero function	The x-axis
0	$f(x) = a_0, \quad a_0 \neq 0$	Constant function	Horizontal line with y-intercept a_0
1	$f(x) = a_1x + a_0, \quad a_1 \neq 0$	Linear function	Nonvertical, nonhorizontal line with slope a_1 and y-intercept a_0
2	$f(x) = a_2x^2 + a_1x + a_0, \quad a_2 \neq 0$	Quadratic function	Parabola: Graph opens up if $a_2 > 0$; graph opens down if $a_2 < 0$



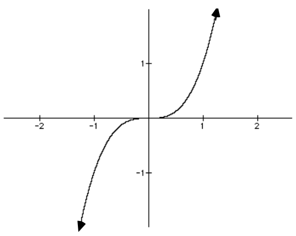
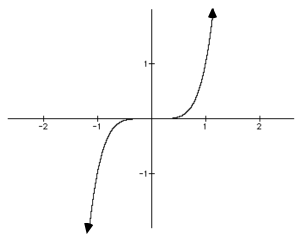
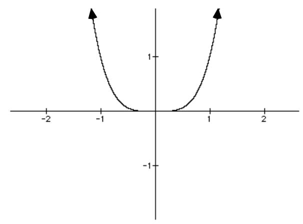
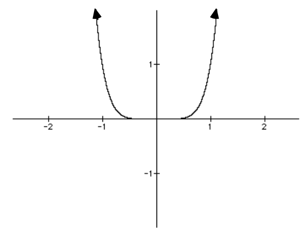
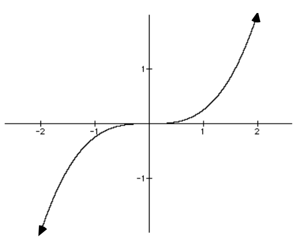
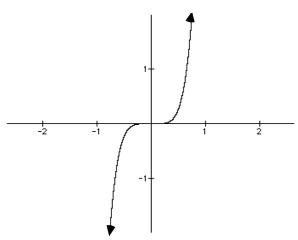
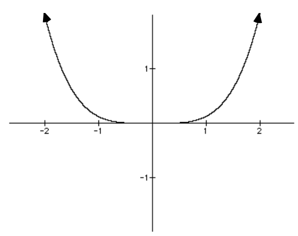
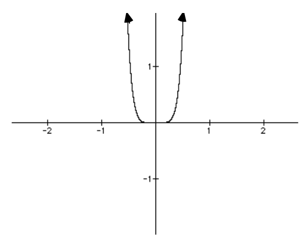
(a) Graph of a polynomial function: smooth, continuous



(b) Cannot be the graph of a polynomial function

Polynomials are **continuous** (no breaks in the graph) and **smooth** (no sharp angles, only rounded curves)

Graphing Functions of the Form: $P(x) = ax^n$

$P(x) = x^3$	$P(x) = x^5$	$P(x) = x^4$	$P(x) = x^6$
			
			
$P(x) = \frac{1}{3}x^3$	$P(x) = 8x^5$	$P(x) = \frac{1}{8}x^4$	$P(x) = 9x^6$

<u>Note</u>: The graph of $y = x^n$ is similar to the graph of

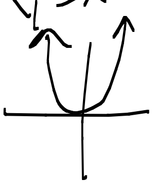
$\begin{cases} y = x^2 \text{ if n is even} \\ y = x^3 \text{ if n is odd} \end{cases}$, except that the greater n is, the flatter the graph is on [-1, 1] and the steeper it is on $(-\infty, -1) \cup (1, \infty)$.

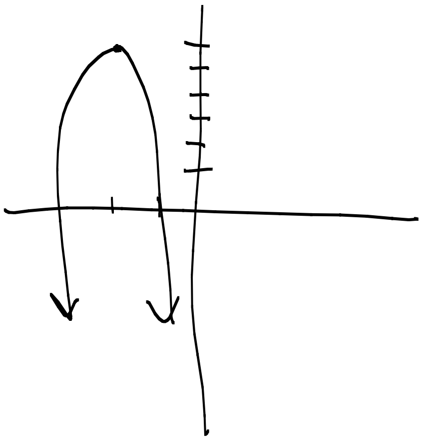
Examining Vertical and Horizontal Translations (Shifts):

Example 1: Graph

a.) $y = -(x + 2)^4 + 6$

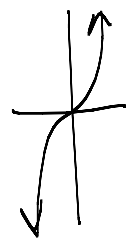
$y = x^4$



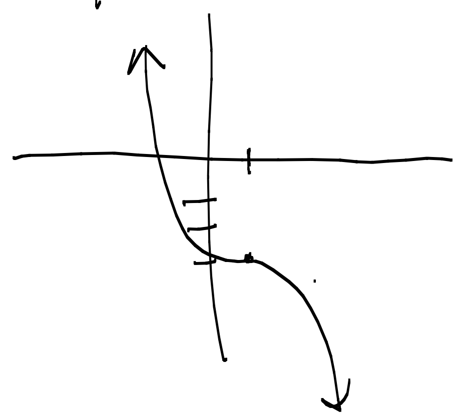


b.) $y = -3 - (x - 1)^3$

$y = x^3$



$y = -(x-1)^3 - 3$



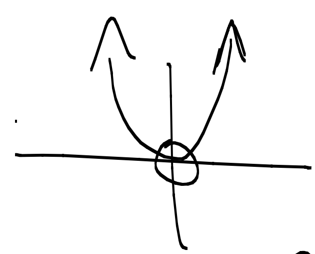
Finding a polynomial from its Zeros:

Example Find a polynomial of degree 3 whose zeros are -4, -2, and 3.

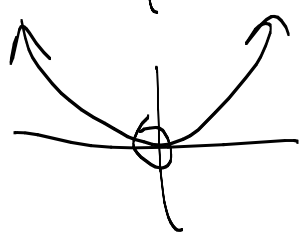
$x = -4 \quad x = -2 \quad x = 3$

$(x+4) \quad (x+2) \quad (x-3)$

$$\boxed{y = a(x+4)(x+2)(x-3)}$$



$y = 1x^2$



$y = 3x^2$

Identifying Zeros and Their Multiplicities

For the polynomial, list all zeros and their multiplicities.

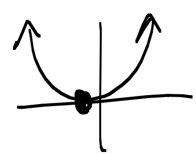
$$f(x) = -2(x-2)^{1}(x+1)^{3}(x-3)^{4}$$

Zeros

$x = +2$ $\quad$ $x = -1$ $\quad$ $x = +3$

mult 3 $\quad$ mult 4

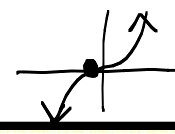
$\boxed{y = x^2}$



If r Is a Zero of Even Multiplicity

Sign of $f(x)$ does not change from one side of r to the other side of r. — Graph **touches** x-axis at r.

$\boxed{y = x^3}$



If r Is a Zero of Odd Multiplicity

Sign of $f(x)$ $\boxed{\text{changes}}$ from one side of r to the other side of r. — Graph **crosses** x-axis at r.

Theorem

Turning Points

If f is a polynomial function of degree n, then f has at most $n - 1$ turning points.

If the graph of a polynomial function f has $n - 1$ turning points, the degree of f is at least n.

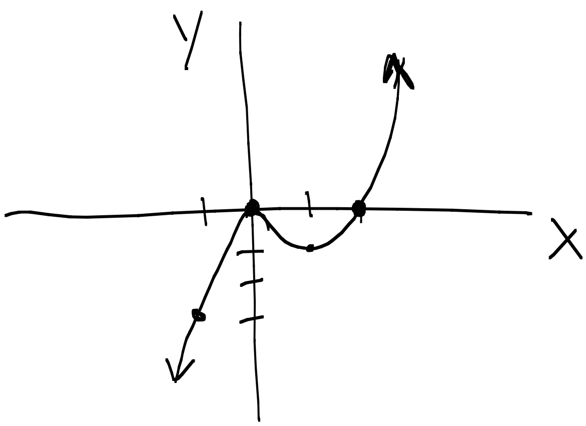
Example Graphing a Polynomial using x-intercepts

For the polynomial: $\boxed{f(x) = x^2(x - 2)}$ factored form → $x^3 - 2x^2 = f(x)$

(a) Find the x- and y-intercepts of the graph of f.

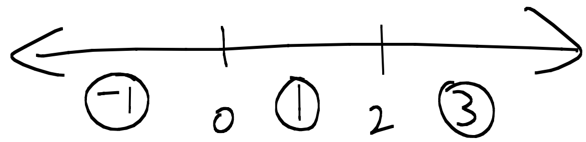
(b) Use the x-intercepts to find the intervals on which the graph of f is above the x-axis and the intervals on which the graph of f is below the x-axis.

(c) Locate other points on the graph and connect all the points plotted with a smooth, continuous curve.



$X = 0 \quad X = 2$

$X = 0$



$x^2(x-2)$

$(-1)^2(-1-2) = 1(-3) = -3$

	0		2	x
Interval	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
Number Chosen	-1	1	3
Value of *f*	$f(-1) = -3$	$f(1) = -1$	$f(3) = 9$
Location of Graph	Below x-axis	Below x-axis	Above x-axis
Point on Graph	$(-1, -3)$	$(1, -1)$	$(3, 9)$

§4.2 Properties of Rational Functions

Rational Function - a function of the form

$$f(x) = \frac{p(x)}{q(x)}$$

where p(x) and q(x) are polynomials with $q(x) \neq 0$.

Find the Domain of a Rational Function

The domain is the set of all real numbers where the denominator $\neq 0$.

(a) The domain of $R(x) = \dfrac{2x^2 - 4}{x + 5}$ is the set of all real numbers x except -5; that is, $\{x \mid x \neq -5\}$.

(b) The domain of $R(x) = \dfrac{1}{x^2 - 4}$ is the set of all real numbers x except -2 and 2, that is, $\{x \mid x \neq -2, x \neq 2\}$.

$x^2 - 4 = 0 \quad x^2 = 4$

(c) The domain of $R(x) = \dfrac{x^3}{x^2 + 1}$ is the set of all real numbers.

$\mathbb{R}$

$x^2 + 1 = 0 \quad \sqrt{x^2} = \sqrt{-1} \quad \cancel{x = \pm i}$

(d) The domain of $R(x) = \dfrac{-x^2 + 2}{3}$ is the set of all real numbers.

(e) The domain of $R(x) = \dfrac{x^2 - 1}{x - 1}$ is the set of all real numbers x except 1, that is, $\{x \mid x \neq 1\}$.

$\cancel{\dfrac{(x+1)(x-1)}{x-1} = x+1}$

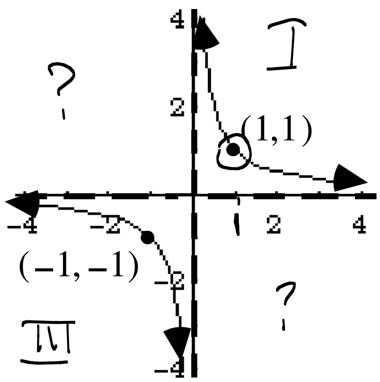
It is important to observe that the functions

$$R(x) = \frac{x^2 - 1}{x - 1} \quad \text{and} \quad f(x) = x + 1$$

are not equal, since the domain of R is $\{x \mid x \neq 1\}$ and the domain of f is the set of all real numbers.

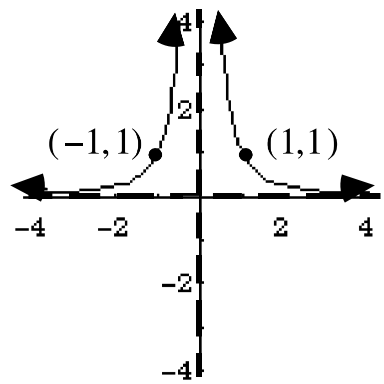
The graphs of rational functions approach (get closer and closer to) lines called **<u>asymptotes</u>**.

Basic Graphs (memorize these)



$$\frac{1}{\left(\frac{1}{2}\right)} = 2$$

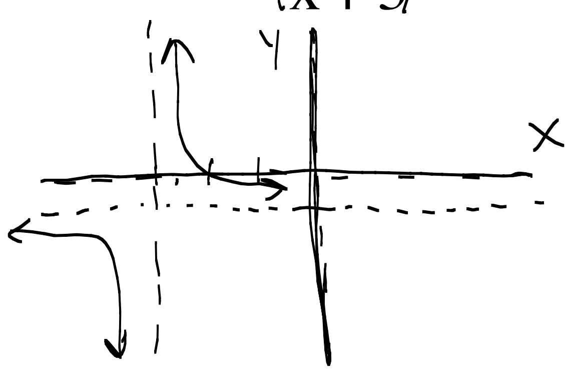
$$\frac{1}{x} \neq 0 \, ?$$



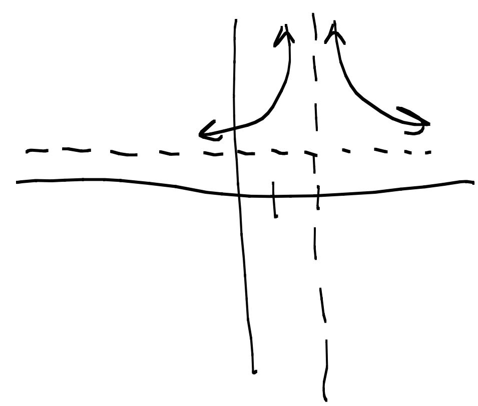
$$\boxed{y = \frac{1}{x}} \qquad \boxed{y = \frac{1}{x^2}} \neq 0$$

Example 1 Use stretching/shrinking, reflecting and shifting rules to graph the following.

a.) $f(x) = \dfrac{1}{(x+3)} - 1$



b.) $f(x) = \dfrac{1}{(x-2)^2} + 1$



VA -When in lowest terms !

To Find the Asymptotes of a Rational Function:

(1) Vertical Asymptotes - Find any vertical asymptotes by setting the denominator equal to 0 and solving for x to get the equation $\boxed{x = a}$.

(2) Horizontal Asymptotes

bottom heavy

Rule 1: If the numerator has lower degree than the denominator, the horizontal asymptote is $\boxed{y = 0}$.

Rule 2: If the numerator and denominator have the same degree and a_n is the leading coefficient of the numerator and b_n is the leading coefficient of the denominator, the horizontal asymptote is $\boxed{y = \frac{a_n}{b_n}}$.

Rule 3: If the numerator has higher degree than the denominator, there is **no horizontal asymptote**.

(rare) Top heavy

(3) Slant Asymptotes - If the numerator is of degree exactly one more than the denominator, there is an slant asymptote. To find it, divide the numerator by the denominator and disregard any remainder. The equation of the slant asymptote is the result of setting y = to the quotient.

Example 2 Give the equations of the vertical, horizontal and/or slant asymptotes of the rational function.

a.) $f(x) = \dfrac{3x}{(x+1)(x-2)}$

V.A. Set D = 0
$(x+1)(x-2)=0$
$x=-1$ $x=+2$

H.A. bottom heavy
N → 1
D → 2
$y=0$

S.A. none

b.) $f(x) = \dfrac{(x-5)(x-2)}{x^2+9}$

V.A.
$x^2+9=0$
$\sqrt{x^2}=\sqrt{-9}$
$\cancel{x=\pm 3i}$
none

H.A.
N → 2 Same
D → 2
$y=\frac{a_n}{b_n}=\frac{1}{1}$
$y=1$

S.A. None

$6x^2$

c.) $f(x) = \dfrac{2x^2+3}{x-4}$

V.A.
$x-4=0$
$x=4$

H.A.
N → 2
D → 1
top heavy
none

S.A.
$$\begin{array}{r|l} & 2x+8 \\ x-4 & 2x^2+0x+3 \\ & -(2x^2-8x) \\ & 8x+3 \\ & -(8x-32) \\ & 35 \end{array}$$
$y=2x+8$

d.) $f(x) = \dfrac{2(3x-1)(x+4)}{(x+2)(5x-3)}$

V.A.
$(x+2)(5x-3)=0$
$x=-2$ $x=\frac{3}{5}$

H.A.
N → 2 Same
D → 2
$y=\frac{a_n}{b_n}=\frac{6}{5}$

S.A. none

Example 3 Find the x-intercepts and y-intercept of the rational function.

a.) $f(x) = \dfrac{3x}{(x+1)(x-2)} = 0$

y int
$x=0$
$f(0)=\frac{3(0)}{(0+1)(0-2)}=\frac{0}{-2}=0$

x int = set N = 0
$y=0$
$\frac{3x}{3}=\frac{0}{3}$
$x=0$

b.) $f(x) = \dfrac{(x-5)(x-2)}{x^2+9}$

y int
$x=0$
$f(0)=\frac{(0-5)(0-2)}{0^2+9}=\frac{10}{9}$

x int
$y=0$
$f(x)=0$
$N=0$
$(x-5)(x-2)=0$
$x=5$ $x=2$

§4.3 Graphs of Rational Functions

Guidelines for Graphing Rational Functions

1. Find and plot the x-intercepts.
   (Set numerator = 0 and solve for x)
2. Find and plot the y-intercepts.
   (Let x = 0 and solve for y)
3. Find and plot the Vertical Asymptotes.
   (Set denominator = 0 and solve for x)
4. Find and plot the Horizontal Asymptotes.
   none y=0
   (Top heavy, Bottom heavy or Same) $y = \frac{a_n}{b_n}$
5. Find and plot the Slant Asymptotes. Rare
   (Divide numerator by denominator.)
6. Find where the graph will intersect its nonvertical asymptote by solving f(x) = k, where k is the y-value of the horizontal asymptote, or f(x) = mx + b, where y = mx + b is the equation of the oblique asymptote. Rare
7. Plot at least one point between and beyond each x-intercept and vertical asymptotes. calc

Use smooth curves to complete the graph between and beyond the vertical asymptotes.

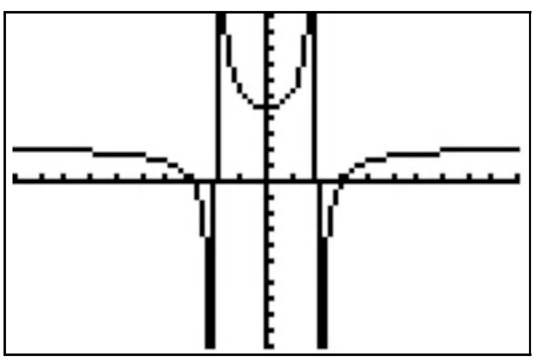
Examples Sketch the graph and provide information about intercepts and asymptotes.

a.) $f(x) = \frac{2(x^2 - 9)}{x^2 - 4}$

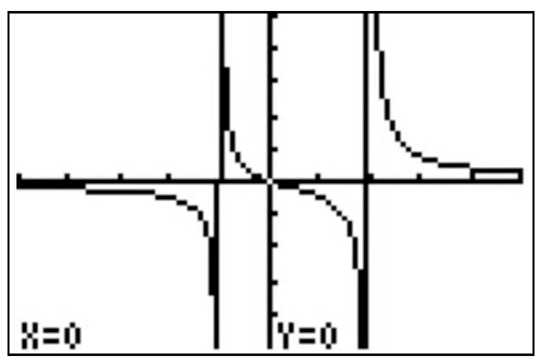
b.) $f(x) = \frac{x}{x^2 - x - 2}$

c.) $f(x) = \frac{x^2 - x - 2}{x - 1}$

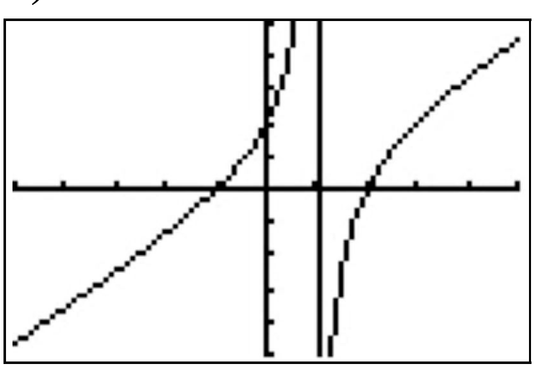
a)



b)



c)



Guidelines for Graphing Rational Functions

example a.) $f(x) = \dfrac{2(x^2 - 9)}{x^2 - 4}$

1. Find and plot the x-intercepts. (Set numerator = 0 and solve for x).

$2(x^2 - 9) = 0$

$x^2 - 9 = 0$

$x^2 = 9$

$x = \pm 3$

---

2. Find and plot the y-intercepts. (Let x = 0 and solve for y)

$f(0) = \dfrac{2(0^2 - 9)}{0^2 - 4} = \dfrac{9}{2}$ $\dfrac{-18}{-4}$

---

3. Find and plot the Vertical Asymptotes. (Set denominator = 0 and solve for x)

$x^2 - 4$

$x^2 = 4$

$x = \pm 2$

---

4. Find and plot the Horizontal Asymptotes. (Top heavy, Bottom heavy or Same)

$f(x) = \dfrac{2(x^2 - 9)}{x^2 - 4}$ Rule 2 Numerator and denominator have the <u>same degree.</u>

$y = 2$ H.A.

---

5. Find and plot the Slant Asymptotes. (Divide numerator by denominator.)

None ! Only have these if Numerator is exactly 1 degree higher than denominator!

---

6. Find where the graph will intersect its nonvertical asymptote by solving f(x) = k, where k is the y-value of the horizontal asymptote, or f(x) = mx + b, where y = mx + b is the equation of the oblique asymptote.

$\cancel{2}x^2 - 8 = \cancel{2}x^2 - 18$

$-8 = -18$

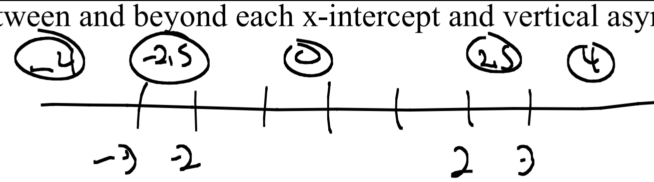
Solve $\dfrac{2}{1} = \dfrac{2(x^2 - 9)}{x^2 - 4}$ (No solution!) No oblique asymptotes.

---

7. Plot at least one point between and beyond each x-intercept and vertical asymptotes.

Remember Test Points ?

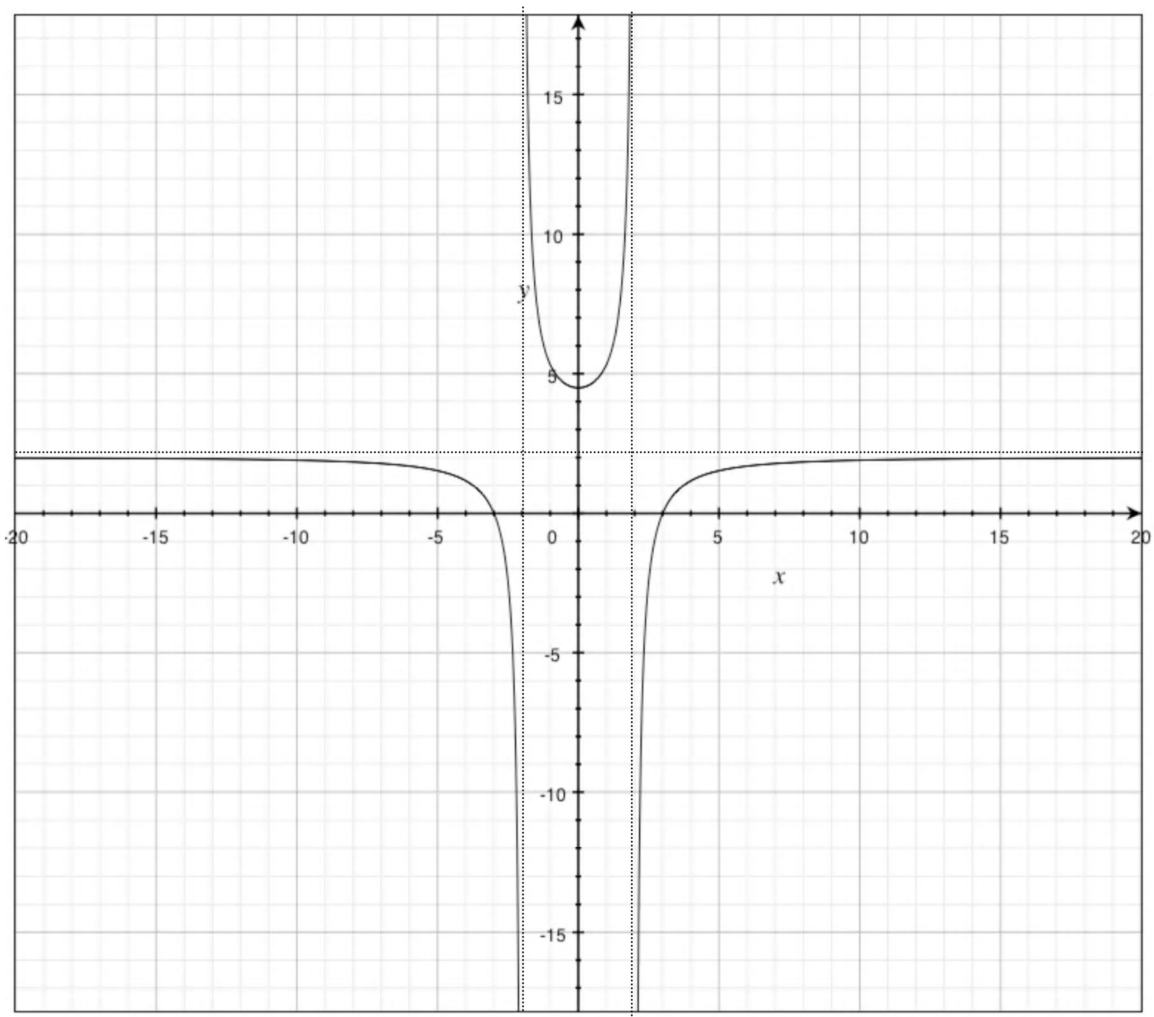
Choose test points carefully!



x = -4	x = -2.5	x = 0	x = 2.5	x = 4			x = -1	x = 1	
y = 1.16	y = -2.4	y = 4.5	y = -2.4	y = 1.16			y = 5.3	y = 5.3	

Note: YOU STILL MAY HAVE TO PLOT ADDITIONAL POINTS !

Use smooth curves to complete the graph between and beyond the vertical asymptotes.


y
15
10
5
-5
-10
-15
x
-20
-15
-10
-5
0
5
10
15
20

!

Example Sketch the graph and provide information about intercepts and asymptotes.

$$f(x) = \frac{x}{x^2 - x - 2}$$

1. Find and plot the x-intercepts. (Set numerator = 0 and solve for x)

$x = 0$

---

2. Find and plot the y-intercepts. (Let x = 0 and solve for y)

$$f(0) = \frac{0}{0^2 - 0 - 2} = 0$$

---

3. Find and plot the Vertical Asymptotes. (Set denominator = 0 and solve for x)

$$x^2 - x - 2 = 0$$
$$(x + 1)(x - 2) = 0$$
$$x = -1 \text{ and } x = 2$$

---

4. Find and plot the Horizontal Asymptotes. (Top heavy, Bottom heavy or Same)

(Rule 1) $y = 0$

---

5. Find and plot the Slant Asymptotes. (Divide numerator by denominator.)

None

---

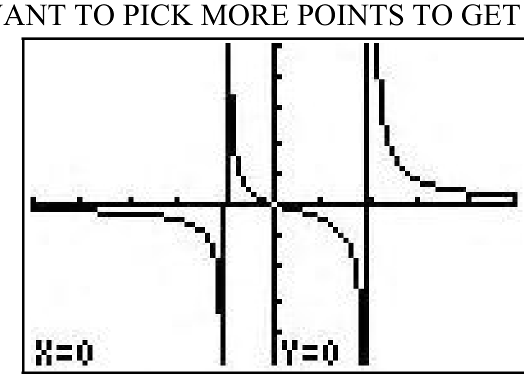
6. Plot at least one point between and beyond each x-intercept and vertical asymptotes.

choose:

x = -2 y = -.5	x = -.5 y = .4	x = 1 y = -.5	x = 3 y = .75

Note: YOU MAY WANT TO PICK MORE POINTS TO GET A BETTER GRAPH !

ANSWER:



Example Sketch the graph and provide information about intercepts and asymptotes.

$$f(x) = \frac{x^2 - x - 2}{x - 1}$$

1. Find and plot the x-intercepts. (Set numerator = 0 and solve for x)

$x^2 - x - 2 = 0$
$(x + 1)(x - 2) = 0$
$x = -1$ and $x = 2$

---

2. Find and plot the y-intercepts. (Let x = 0 and solve for y)

$$f(0) = \frac{0^2 - 0 - 2}{0 - 1} = \frac{-2}{-1} = 2$$

---

3. Find and plot the Vertical Asymptotes. (Set denominator = 0 and solve for x)

$(x - 1) = 0$ $\quad\quad x = 1$

---

4. Find and plot the Horizontal Asymptotes. (Top heavy, Bottom heavy or Same)

(Rule 3) Top Heavy none !

---

5. Find and plot the Slant Asymptotes. (Divide numerator by denominator.)

$x - 1 \overline{)\,x^2 - x - 2}$ $\quad\quad y = x$

$\underline{\quad -x^2 + x}$

$\quad\quad 0$

---

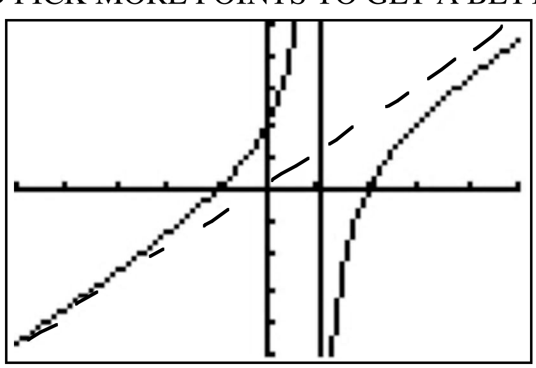
6. Plot at least one point between and beyond each x-intercept and vertical asymptotes.

choose:

x = -2	x = 0	x = 1.5	x = 3
y = -1.3	y = 2	y = -2.5	y = 2

Note: YOU MAY WANT TO PICK MORE POINTS TO GET A BETTER GRAPH !

ANSWER:



Guidelines for Graphing Rational Functions

example a.) $f(x) = \dfrac{2(x^2 - 9)}{x^2 - 4}$

1. Find and plot the x-intercepts. (Set numerator = 0 and solve for x).

$2(x^2 - 9) = 0$

$x^2 - 9 = 0$

$x^2 = 9$

$x = \pm 3$

---

2. Find and plot the y-intercepts. (Let x = 0 and solve for y)

$f(0) = \dfrac{2(0^2 - 9)}{0^2 - 4} = \dfrac{9}{2}$

---

3. Find and plot the Vertical Asymptotes. (Set denominator = 0 and solve for x)

$x^2 - 4$

$x^2 = 4$

$x = \pm 2$

---

4. Find and plot the Horizontal Asymptotes. (Top heavy, Bottom heavy or Same)

$f(x) = \dfrac{2(x^2 - 9)}{x^2 - 4}$ Rule 2 Numerator and denominator have the <u>same degree.</u>

$y = 2$ H.A.

---

5. Find and plot the Slant Asymptotes. (Divide numerator by denominator.)

None ! Only have these if Numerator is exactly 1 degree higher than denominator!

---

6. Find where the graph will intersect its nonvertical asymptote by solving f(x) = k, where k is the y-value of the horizontal asymptote, or f(x) = mx + b, where y = mx + b is the equation of the oblique asymptote.

Solve $2 = \dfrac{2(x^2 - 9)}{x^2 - 4}$ (No solution!) No oblique asymptotes.

---

7. Plot at least one point between and beyond each x-intercept and vertical asymptotes.

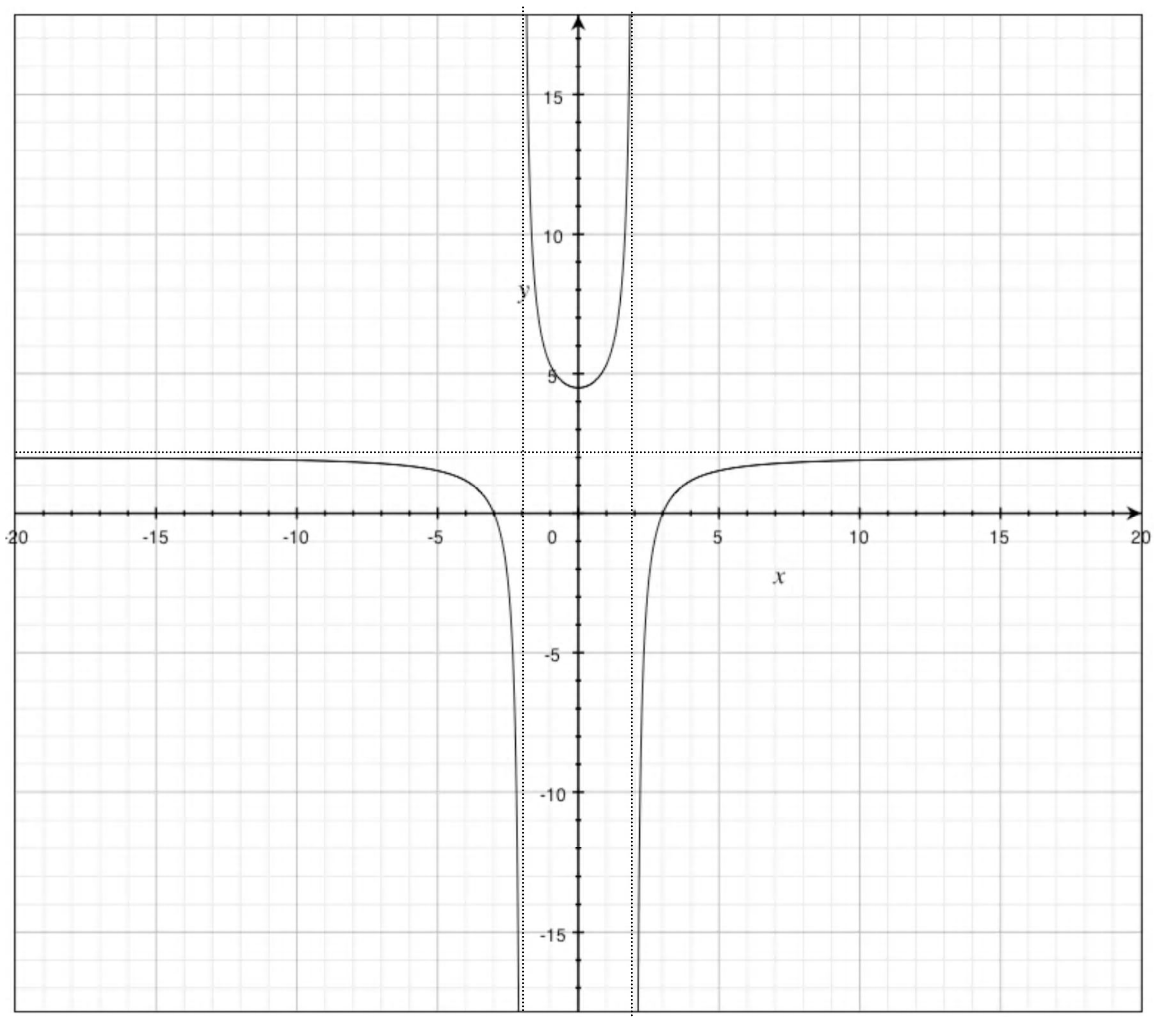
Remember Test Points ?

Choose test points carefully!

x = -4	x = -2.5	x = 0	x = 2.5	x = 4			x = -1	x = 1	
y = 1.16	y = -2.4	y = 4.5	y = -2.4	y = 1.16			y = 5.3	y = 5.3	

Note: YOU STILL MAY HAVE TO PLOT ADDITIONAL POINTS !

Use smooth curves to complete the graph between and beyond the vertical asymptotes.


15
10
5
-5
-10
-15
y
-20
-15
-10
-5
0
5
10
15
20
x
!

Example Sketch the graph and provide information about intercepts and asymptotes.

$$f(x) = \frac{x}{x^2 - x - 2}$$

1. Find and plot the x-intercepts. (Set numerator = 0 and solve for x)

$x = 0$

---

2. Find and plot the y-intercepts. (Let x = 0 and solve for y)

$$f(0) = \frac{0}{0^2 - 0 - 2} = 0$$

---

3. Find and plot the Vertical Asymptotes. (Set denominator = 0 and solve for x)

$x^2 - x - 2 = 0$
$(x + 1)(x - 2) = 0$
$x = -1$ and $x = 2$

---

4. Find and plot the Horizontal Asymptotes. (Top heavy, Bottom heavy or Same)

(Rule 1) $y = 0$

---

5. Find and plot the Slant Asymptotes. (Divide numerator by denominator.)

None

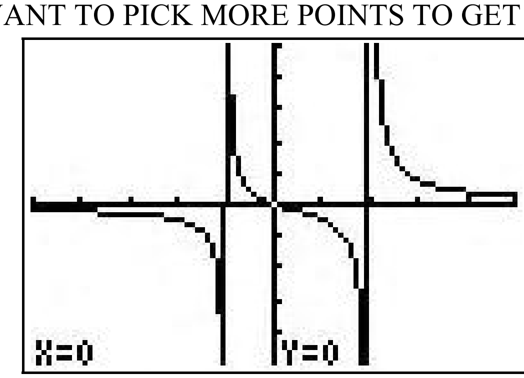
---

6. Plot at least one point between and beyond each x-intercept and vertical asymptotes.

choose:

x = -2 y = -.5	x = -.5 y = .4	x = 1 y = -.5	x = 3 y = .75

Note: YOU MAY WANT TO PICK MORE POINTS TO GET A BETTER GRAPH !

ANSWER: 

Example Sketch the graph and provide information about intercepts and asymptotes.

$$f(x) = \frac{x^2 - x - 2}{x - 1}$$

1. Find and plot the x-intercepts. (Set numerator = 0 and solve for x)

$x^2 - x - 2 = 0$
$(x + 1)(x - 2) = 0$
$x = -1$ and $x = 2$

---

2. Find and plot the y-intercepts. (Let x = 0 and solve for y)

$$f(0) = \frac{0^2 - 0 - 2}{0 - 1} = \frac{-2}{-1} = 2$$

---

3. Find and plot the Vertical Asymptotes. (Set denominator = 0 and solve for x)

$(x - 1) = 0$ $x = 1$

---

4. Find and plot the Horizontal Asymptotes. (Top heavy, Bottom heavy or Same)

(Rule 3) Top Heavy none !

---

5. Find and plot the Slant Asymptotes. (Divide numerator by denominator.)

$$x - 1\overline{\smash{\big)}\,x^2 - x - 2} \qquad y = x$$
$$\underline{\quad -x^2 + x}$$
$$0$$

---

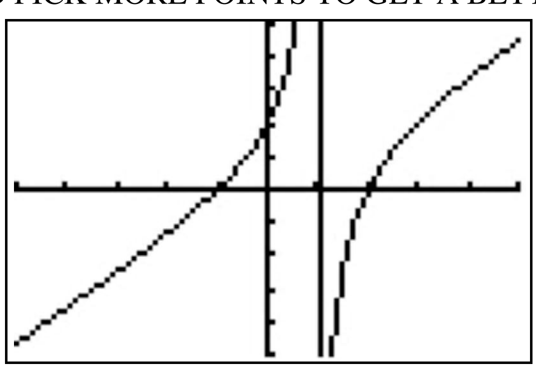
6. Plot at least one point between and beyond each x-intercept and vertical asymptotes.

choose:

x = -2 y = -1.3	x = 0 y = 2	x = 1.5 y = -2.5	x = 3 y = 2

Note: YOU MAY WANT TO PICK MORE POINTS TO GET A BETTER GRAPH !

ANSWER:



§4.4 Polynomial and Rational Inequalities

Steps for Solving Polynomial and Rational Inequalities

Step 1: Write the inequality so that a polynomial or rational expression f is on the left side and zero is on the right side in one of the following forms:

$$f(x) > 0 \qquad f(x) \geq 0 \qquad f(x) < 0 \qquad f(x) \leq 0$$

For rational expressions, be sure that the left side is written as a single quotient.

Step 2: Determine the numbers at which the expression f on the left side equals zero and, if the expression is rational, the numbers at which the expression f on the left side is undefined. critical numbers

Step 3: Use the numbers found in Step 2 to separate the real number line into intervals.

Step 4: Select a number in each interval and evaluate f at the number.

(a) If the value of f is positive, then $f(x) > 0$ for all numbers x in the interval.

(b) If the value of f is negative, then $f(x) < 0$ for all numbers x in the interval.

If the inequality is not strict, include the solutions of $f(x) = 0$ in the solution set.

Rational Inequalities:

Note:– **NEVER** multiply both sides of an inequality by a variable expression!!

- You cannot lose the denominator in quotients.
- Always remember the restriction that the denominator cannot be zero.

Examples Solve.

a.) $x^4 \le 4x^2$

$\boxed{x^4 - 4x^2 \le 0}$

$x^2(x^2 - 4) = 0$

$x^2 = 0$ | $x^2 - 4 = 0$

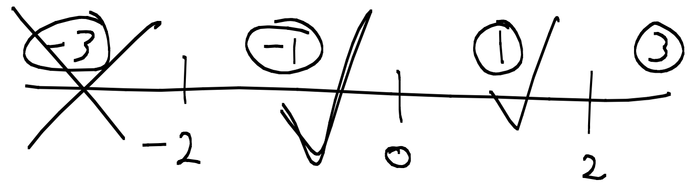
$x = 0$ | $x = 0$

$x^2 = 4$

$x = \pm 2$

critical numbers

$[-2, 2]$



test pts

$x = -3$

$(-3)^4 - 4(-3)^2 \le 0?$

$81 - 36$

$45 \le 0?$ no!

$x = -1$

$y = -3 \le 0?$

yes

$x = 1$

$y = -3 \le 0?$

yes

$x = 3$

$y = 45 \le 0?$

no

b.) $x^4 > x$

① $\boxed{x^4 - x > 0}$

② $x^4 - x = 0$

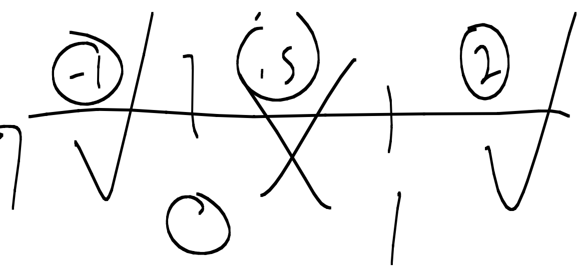
$x(x^3 - 1) = 0$

$x(x^3 - (1)^3) = 0$

$x(x - 1)\cancel{(x^2 + x + 1)} = 0$

$x = 0$ $x = 1$ critical #'s

$\boxed{(-\infty, 0)\ (1, \infty)}$



test pts

$x = -1$

$y = +2 > 0?$

yes

$x = .5$

$y = -.4375 > 0?$

no

$x = 2$

$y = 14 > 0?$

yes

Single Fraction

c.) $\boxed{\dfrac{(x+3)(2-x)}{(x-1)^2} > 0}$

1) SF | SF

~~$(-3, 2)$~~

2) $(x+3)(2-x) = 0$

$x+3=0$ $\quad$ $2-x=0$

$x=-3$ $\quad$ $x=2$

$(x-1)^2 = 0$

$x-1=0$

$x=1$ $\quad$ $\boxed{(-3,1)\cup(1,2)}$

$\left(\frac{x}{2}\right)$ $\quad$ ~~$\frac{x}{4} + \frac{x}{7}$~~

3) -4 $\quad$ 0 $\quad$ 1.5 $\quad$ 3

$-3 \quad 1 \quad 2$

test pt

$x=-4$

$y = f(x) = -.24$

no

$x = 1.5$

$y = 9 > 0$ yes!

$x=0$

$y = 6 > 0$ yes!

$x = 3$

$y = -1.5 > 0$

no!

d.) $\dfrac{4x+5}{x+2} \geq 3$ $\qquad$ $\dfrac{4x+5}{x+2} - \dfrac{3(x+2)}{1(x+2)} \geq 0$

$\dfrac{4x+5-3x-6}{x+2} \geq 0$

$-3 \quad 0 \quad 2$

$-2 \quad 1$

Single Frac

$\boxed{\dfrac{(x-1)}{(x+2)} \geq 0}$ $\qquad$ $\boxed{x \neq -2}$

test pts

$x=-3$

$y = 4 \geq 0$? yes

$x=0$

$y = -.5 \geq 0$

no!

$x-1=0$ $\quad$ $x+2=0$

$x=1$ $\quad$ $x=-2$ $\quad$ $\boxed{(-\infty,-2)\cup[1,\infty)}$

$x=2$

$y = .25 \geq 0$ yes

§4.5 Real Zeros of Polynomial Functions

Division Algorithm :

For any polynomial P(x) and any complex number d(x)!, there exists a unique polynomial Q(x) and number r(x) such that:

$$P(x) = d(x) * Q(x) + r(x).$$

$148 = (3 \cdot 49) + 1$

$$\begin{array}{r} 49 \\ 3\overline{)148} \\ -12 \\ \hline 28 \\ -27 \\ \hline 1 \end{array}$$

Example 1: Divide

a) $6q^3 - 17q^2 + 22q - 23$ by $2q - 3$

$$\begin{array}{r} 3q^2 - 4q + 5 \\ 2q-3\overline{)6q^3 - 17q^2 + 22q^1 - 23} \\ -(6q^3 - 9q^2) \\ \hline -8q^2 + 22q \\ -(-8q^2 + 12q) \\ \hline 10q - 23 \\ -(10q - 15) \\ \hline -8 \end{array}$$

$$3q^2 - 4q + 5 + \frac{-8}{2q-3}$$

b) $3x^3 - 2x^2 - 150$ by $1x - 4$

Synthetic div

$1x - k$

Synthetic Division :

$\boxed{1x^1 - k}$ K? x−4

$$\begin{array}{r} 3x^2+10x+40 \\ x-4\overline{)3x^3-2x^2+0x-150} \\ (-)\ \underline{3x^3-12x^2} \\ 10x^2+0x \\ (-)\ \underline{10x^2-40x} \\ 40x-150 \\ (-)\ \underline{40x-160} \\ 10 \end{array}$$ Answer: $3x^2+10x+40+\dfrac{10}{x-4}$	$$\begin{array}{r|rrrr} 4 & 3 & -2 & 0 & -150 \\ & & 12 & 40 & 160 \\ \hline & 3 & 10 & 40 & 10 \end{array}$$ Answer: $\boxed{3x^2+10x+40+\dfrac{10}{x-4}}$

$\underline{x-1}$ $k=+1$

$\underline{x+2}$ $k=-2$

$\underline{x-5}$ $k=+5$

$K=-3$

Example 2: Divide by synthetic division.

$x^4 - 10x^2 - 2x + 4$ by $x + 3$

$$\begin{array}{r|rrrrr} -3 & 1 & 0 & -10 & -2 & 4 \\ & & -3 & 9 & +3 & -3 \\ \hline & 1 & -3 & -1 & 1 & 1 \end{array}$$

$$\begin{array}{r} 81 \\ -90 \\ 6 \\ +4 \\ \hline 1 \end{array}$$

$$\boxed{x^3 - 3x^2 - x + 1 + \frac{1}{x+3}}$$

The Remainder Theorem

If a polynomial f(x) is divided by x – k, the remainder is equal to f(k).

Example 3: Find the remainder if $f(x) = x^3 - 4x^2 - 5$ is divided by a) x – 3 b) x + 2

a) $k = +3$

$f(3) = (3)^3 - 4(3)^2 - 5$

$= -14$

b) $k = -2$

$f(-2) = (-2)^3 - 4(-2)^2 - 5$

$= -8 - 16 - 5$

$= -29$

The Factor Theorem

The polynomial x – k is a factor of the polynomial f(x) if and only if f(k) = 0.

Example 4: Use the Factor Theorem to determine whether the function $f(x) = 2x^3 - x^2 + 2x - 3$ has the factor

a) x – 1 $k = +1$

yes

$f(1) = 2 - 1 + 2 - 3$

$= 0$

$$\begin{array}{r|rrrr} 1 & 2 & -1 & 2 & -3 \\ & & 2 & 1 & 3 \\ \hline & 2 & 1 & 3 & 0 \end{array}$$

b) x + 3 $k = -3$

no

$f(-3) = -54 - 9 - 6 - 3$

$= -72$

Number of Real Zeros (Theorem)

A polynomial function cannot have more real zeros than its degree.

Descartes' Rule of Signs

Let f denote a polynomial function written in standard form.

(1) The number of positive real zeros of f either equals the number of variations in the sign of the nonzero coefficients of $f(x)$ or else equals that number less an even integer.

(2) The number of negative real zeros of f either equals the number of variations in the sign of the nonzero coefficients of $f(-x)$ or else equals that number less an even integer.

Discuss the real zeros of $3x^6 - 4x^4 + 3x^3 + 2x^2 - x^1 - 3$

+ real 0's
3 sign changes so

3 + real zeros
or
1 + real zeros

- real 0's
3 sign changes

$f(-x) = +3x^6 - 4x^4 - 3x^3 + 2x^2 + x - 3$

$= +3(-x)^6 - 4(-x)^4 + 3(-x)^3 + 2(-x)^2 - (-x)^1 - 3$

3 (-) real zeros
or
1 (-) real zeros

Rational Zeros Theorem:

If the polynomial

p

q

$$f(x) = \boxed{a_n} x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + \boxed{a_0}$$

has integer coefficients, every rational zero of f(x) has the form

$$\text{Rational Zero} = \frac{p}{q}$$

where p and q have no common factors other than 1, and

p is a factor of the constant term $\boxed{a_0}$ and
q is a factor of the leading coefficient a_n

$$\text{Possible rational zeros} = \frac{\text{factors of constant term } a_0}{\text{factors of leading coefficient } a_n}$$

Example 1: List the possible rational zeros for each function

q p

$p = 3 \Rightarrow \pm1, \pm3$

a) $f(x) = 2x^3 + 3x^2 - 8x + 3$

$q = 2 \Rightarrow \pm1, \pm2$

$\frac{p}{q} = \pm1, \pm\frac{1}{2}, \pm3, \pm\frac{3}{2}$

possible

b) $f(x) = 2x^3 + 11x^2 - 7x - 6$

$p: 6 \Rightarrow \pm1, \pm2, \pm3, \pm6$

$q: 2 \Rightarrow \pm1, \pm2$

$\frac{p}{q} = \pm1, \pm\frac{1}{2}, \pm2, \pm3, \pm\frac{3}{2}, \pm6$

Now that we have a list of possible zeros, we need to determine which possible zeros are $\boxed{\text{actual}}$ zeros.

Steps for Finding the Real Zeros of a Polynomial Function

STEP 1: Use the degree of the polynomial to determine the maximum number of zeros.

STEP 2: Use Descartes' Rule of Signs to determine the possible number of positive zeros and negative zeros.

STEP 3: (a) If the polynomial has integer coefficients, use the Rational Zeros Theorem to identify those rational numbers that potentially could be zeros.

(b) Use substitution, synthetic division, or long division to test each potential rational zero.

(c) Each time that a zero (and thus a factor) is found, repeat Step 3 on the depressed equation.

STEP 4: In attempting to find the zeros, remember to use (if possible) the factoring techniques that you already know (special products, factoring by grouping, and so on).

Example : Find all the zeros for the function.

$$f(x) = x^5 - 5x^4 + 12x^3 - 24x^2 + 32x - 16$$

$x=1$	$(x-1)$
$x=2$	$(x-2)$
$x=2$	$(x-2)$
$x=2i$	$(x-2i)$
$x=-2i$	$(x+2i)$

S1) 5 zeros

S2) + real 0's 5 or 3 or 1

− real 0's $f(-x) = -x^5 - 5x^4 - 12x^3 - 24x^2 - 32x - 16$

no − real 0's

S3) $p = 16 \to \pm1, \pm2, \pm4, \pm8, \pm16$

$q = 1 \to \pm1$

$$\begin{array}{r|rrrrrr} 1 & 1 & -5 & 12 & -24 & 32 & -16 \\ & & 1 & -4 & 8 & -16 & 16 \\ \hline & 1 & -4 & 8 & -16 & 16 & 0 \end{array}$$

$$\begin{array}{r|rrrrr} 2 & 1 & -4 & 8 & -16 & 16 \\ & & 2 & -4 & 8 & -16 \\ \hline & 1 & -2 & 4 & -8 & 0 \end{array}$$

$x^3 - 2x^2 + 4x - 8 = 0$

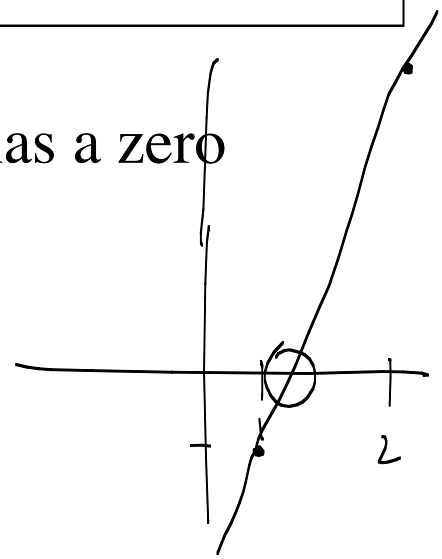
$x^2(x-2) + 4(x-2) = 0$

$x^2+4=0 \quad \sqrt{x^2}=\sqrt{-4} \quad x=\pm 2i \qquad (x^2+4)\underline{(x-2)}=0$

Intermediate Value Theorem:

Let f denote a polynomial function. If a < b and if f(a) y and f(b) are of opposite sign, there is at least one real zero y of f between a and b.

example: Show that $f(x)=x^5-x^3-1$ has a zero between 1 and 2.



$x=1 \qquad f(1)=1-1-1=-1$

$x=2 \qquad f(2)=2^5-2^3-1=23$

§4.6 Complex Zeros; Fundamental Theorem of Algebra

Conjugate Pairs Theorem

Let f(x) be a polynomial whose coefficients are real numbers. If r = a + bi is a zero of f, the complex conjugate $\bar{r}$ = a – bi is also a zero of f.

Corollary

A polynomial f of odd degree with real coefficients has at least one real zero.

Example 1: A polynomial f of degree 5 has zeros 1, 5i, and 1 + i. Find the remaining two zeros.

$x = +1$
$x = 5i \rightarrow 0 + 5i$
$x = -5i$
$x = 1 + i$
$x = 1 - i$

$x = +i$
$x = -i$

$= (x - i)(x + i)$
$= x^2 + \cancel{xi} - \cancel{xi} - i^2$
$= a(x^2 + 1)$

Find the Complex Zeros of a Polynomial

Example: Find the complex zeros of:

$$f(x) = 3x^4 + 5x^3 + 25x^2 + 45x - 18$$

Find the Complex Zeros of a Polynomial

Example: Find the complex zeros of:

$$f(x) = 3x^4 + 5x^3 + 25x^2 + 45x - 18$$

S1) degree $= 4$

S2 + real 0's

① (1 sign change)

− real 0's

$f(-x) = 3x^4 - 5x^3 + 25x^2 - 45x - 18$

1 1 1 0

3 or 1 (3 sign changes)

S3) $p = 18 \rightarrow \pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$

$q = 3 \rightarrow \pm 1, \pm 3$

$\frac{p}{q} = \pm 1, \pm \frac{1}{3}, \pm 2, \pm \frac{2}{3}, \pm 3, \pm 6, \pm 9, \pm 18$

S4) choose ~~$k = 1$~~ or ~~$k = -1$~~ (doesn't work remainder $\neq 0$)

$k = -2$ works

−2	3	5	25	45	−18
		−6	2	−54	18
	3	−1	27	−9	0 ✓

S4) continued

Depressed equation

then $3x^3 - x^2 + 27x - 9 = 0$

so $x^2(3x-1) + 9(3x-1) = 0$

$(x^2+9)(3x-1) = 0$

$3x - 1 = 0$

$x = +\frac{1}{3}$

$x^2 + 9 = 0$

$\sqrt{x^2} = \sqrt{-9}$

$x = \pm 3i$

conjugate pair

zeros!

$x = -2$

$x = \frac{1}{3}$

$x = 3i$

$x = -3i$

factors!

$(x+2)$

$(3x-1)$

$(x-3i)$

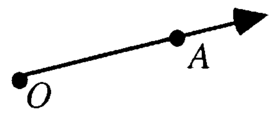
$(x+3i)$

Be sure to Scan!

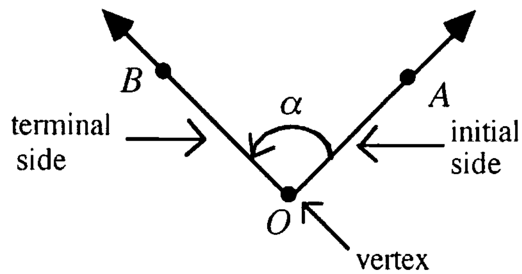
§6.1 Angles and Their Measure

So what does trigonometry mean ? **measurement of triangles!**

- a ray starts at a point and extends indefinitely

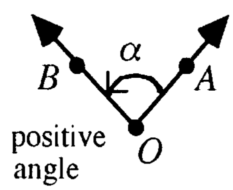


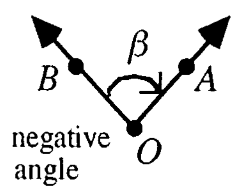
- an angle occurs when a ray is rotated about its endpoint
- the starting position of the ray is the initial side of the angle
- the position of the ray after rotation is the terminal side of the angle
- the meeting point of the two rays is the vertex of the angle



$\angle\alpha = \angle O = \angle AOB$

- a positive angle is formed by a counter-clockwise rotation
- a negative angle is formed by a clockwise rotation
- coterminal angles have the same initial and terminal sides.





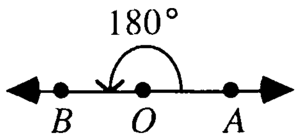
Degree Measure

1 revolution = 360°

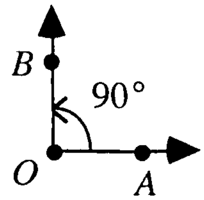
• an angle formed by rotating a ray $\frac{1}{360}$ of a complete revolution has a measure of 1 <u>degree</u> (1°)

• angles are often classified by their measures

(1) a <u>straight angle</u> has a measure of 180°

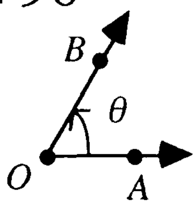


(2) a <u>right angle</u> has a measure of 90°



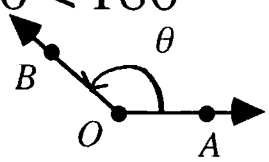
(3) an <u>acute angle</u> has a measure

$0° < \theta < 90°$

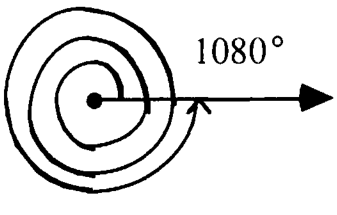


(4) an <u>obtuse angle</u> has a measure

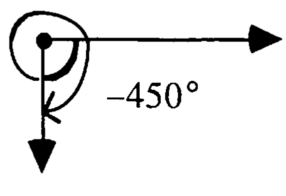
$90° < \theta < 180°$



• angles larger than 360° or smaller than −360° can be measured by considering more than one rotation

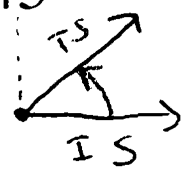


$$\begin{array}{r} 360 \\ 3 \\ \hline 1080 \end{array}$$

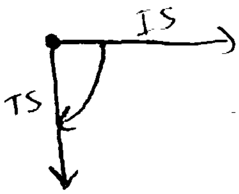


Draw an Angle: (discuss standard position)

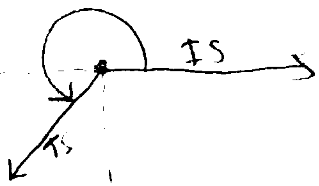
a) 45°



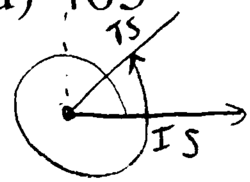
b) −90°

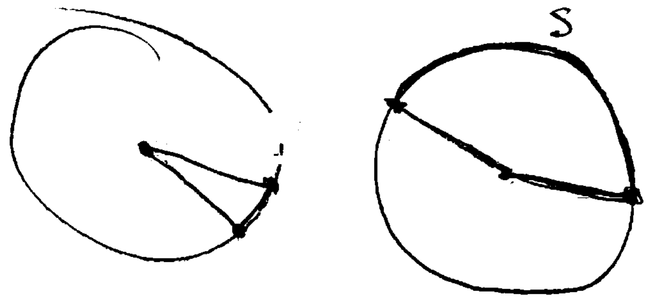


c) 225°

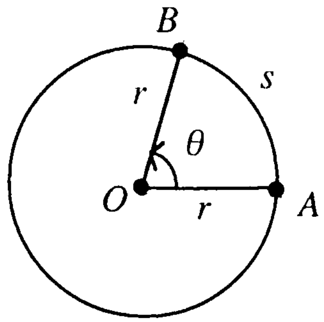


d) 405°




S

Radian Measure



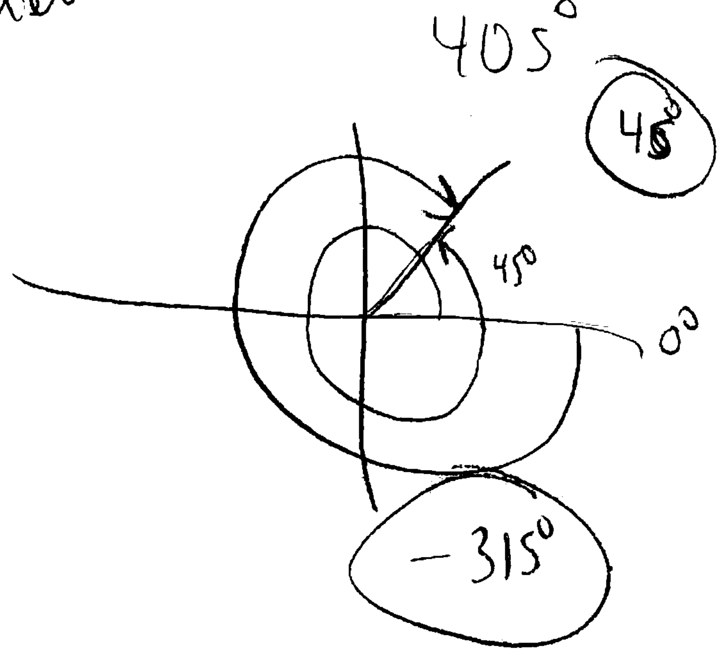
- consider a circle of radius r with two radii OA and OB
- the angle θ formed by these two radii is a central angle
- the arc AB is the part of the circle between A and B and its length is S
- the arc AB subtends the angle θ

- the measure of the central angle subtended by an arc of length r on a circle with radius r is one radian

- the radian measure of the central angle subtended by an arc of length s on a circle of radius r is $\theta = \frac{s}{r}$ or $s = r\theta$ (θ is in radians)

- given a circle of radius r, the radian measure of the central angle subtended by the circumference of the circle is $\theta = \frac{2\pi r}{r} = 2\pi$ while in degrees $\theta = 360°$

- thus, $360° = 2\pi$ radians and $\boxed{180° = \pi}$ radians coterminal?



(radians)

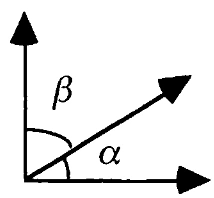
Example: Find the Arc Length of a Circle

Find the length of an arc of a circle of radius 2 meters subtended by a central angle of 0.25 rdian.

radians

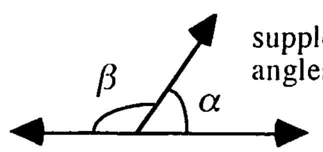
$s = r\theta = (2)(.25) = .5$

- two **nonnegative** angles α and β are <u>complementary angles</u> if $\alpha + \beta = 90°$
- in this case, α is the <u>complement</u> of β and vice versa



complementary angles

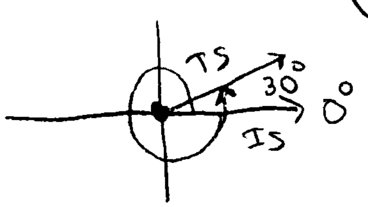
- two **nonnegative** angles α and β are <u>supplementary angles</u> if $\alpha + \beta = 180°$
- in this case, α is the <u>supplement</u> of β and vice versa



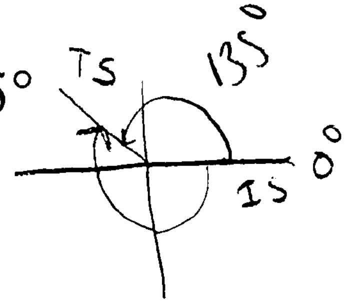
supplementary angles

Example Find the coterminal angles for the following angles.

a) $^{+}390°$ 30°



b) $-225°$



$$30^\circ \cdot \frac{\pi}{180^\circ} = \frac{\pi}{6}$$

Example 2 Find the complement and supplement angles for the following angles.

a) $42°$

comp $= 42° + x = 90$

$x = 48°$

supp $180 - 42 = 138°$

b) $145°$

comp $=$ none

supp $= 180 - 145°$

$= 35°$

Radian-Degree Conversion Factors

- to change radians to degrees, multiply the number of radians by $\dfrac{180°}{\pi}$
- to change degrees to radians, multiply the number of degrees by $\dfrac{\pi}{180°}$

Example Convert from Degrees to Radians.

a) $\dfrac{\cancel{60°}}{1} \cdot \dfrac{\pi}{\cancel{180°}}$

$= \dfrac{\pi}{3}$

b) $\dfrac{-\cancel{45}°}{1} \cdot \dfrac{\pi}{180}$

$= -\dfrac{\pi}{4}$

c) $107°$

$\dfrac{107°}{1} \cdot \dfrac{\pi}{180°} = .6\pi$

1.86 rad

Example Convert from Radians to Degrees

a) $\dfrac{\cancel{\pi}}{6} \cdot \dfrac{180}{\cancel{\pi}}$

$30°$

b) $-\dfrac{3\cancel{\pi}}{2} \cdot \dfrac{\overset{90}{\cancel{180}}}{\cancel{\pi}}$

$= -270°$

c) 3 radians

$\dfrac{3}{1} \cdot \dfrac{180}{\pi} = \dfrac{540}{\pi}$

$= 171.89°$

DMS System (Degree, Minute, Second)

1 minute $(1') = \left(\frac{1}{60}\right)^\circ \quad \Rightarrow 60' = 1^\circ$

1 second $(1'') = \left(\frac{1}{60}\right)' = \left(\frac{1}{3600}\right)^\circ \Rightarrow 60'' = 1'$ and $3600'' = 1^\circ$

Example Convert 50°6'21" to decimal degree measure to the nearest thousandth.

$50^\circ \quad \frac{6'}{60} \quad \frac{21''}{3600}$

$50^\circ + .1 + .0058333$

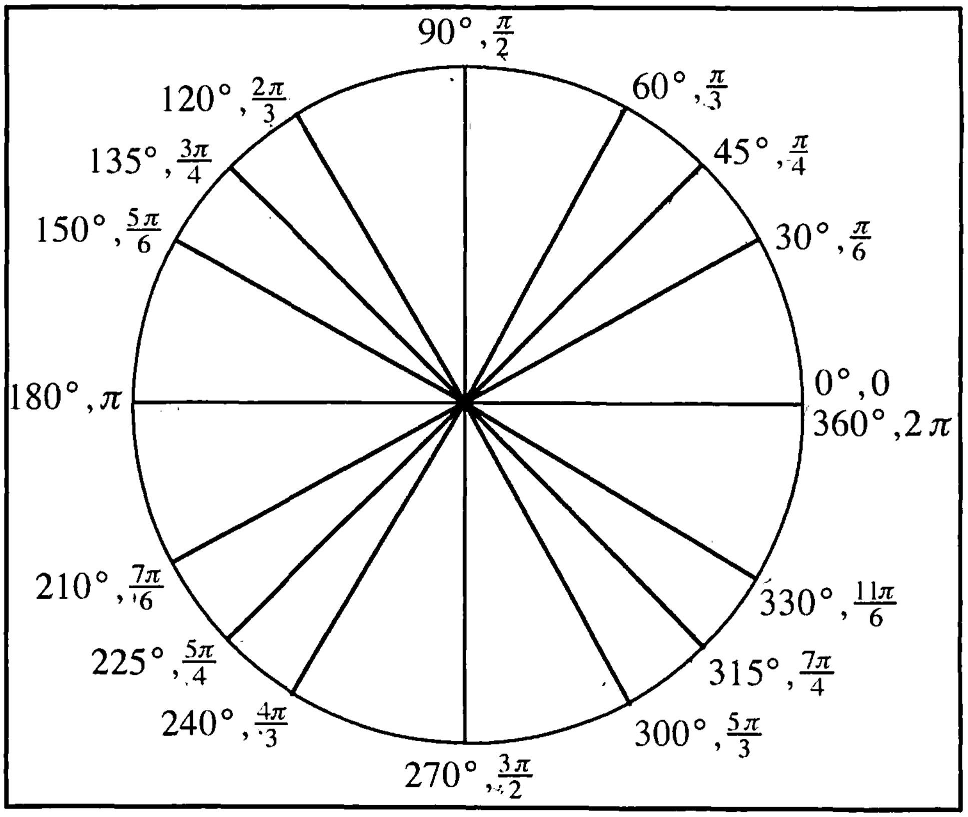
50.1058333°

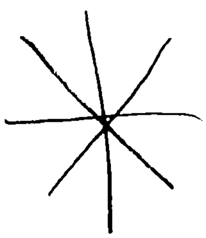
Example Convert 21.256° to DMS.

$21^\circ \; \underline{15}' \; \underline{21.6}''$

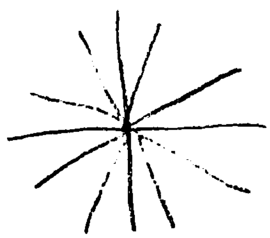
Note: You **MUST** memorize all degree to radian conversions of the selected angles listed below and know their positions on a circle measured from the positive x-axis.

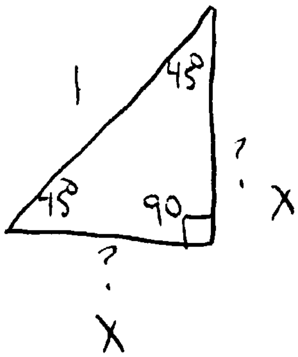
Degrees	**Radians**
0	0
30	$\pi/6$
45	$\pi/4$
60	$\pi/3$
90	$\pi/2$
120	$2\pi/3$
135	$3\pi/4$
150	$5\pi/6$
180	π
210	$7\pi/6$
225	$5\pi/4$
240	$4\pi/3$
270	$3\pi/2$
300	$5\pi/3$
315	$7\pi/4$
330	$11\pi/6$
360	2π

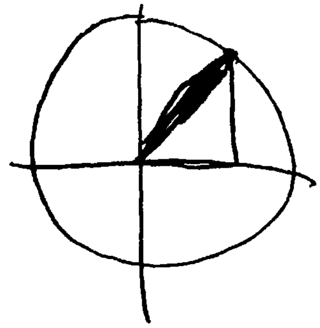




45°







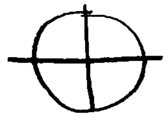
$$x^2 + x^2 = 1^2$$

$$\frac{2x^2}{2} = \frac{1}{2}$$

$$\sqrt{x^2} = \sqrt{\frac{1}{2}}$$

$$x = \pm\frac{\sqrt{1}}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

§6.2 Trigonometric Functions: The Unit Circle

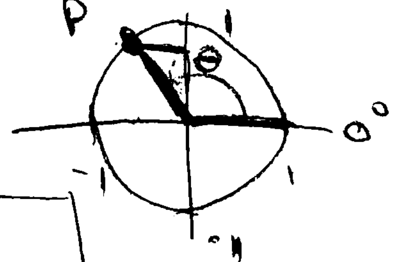
Discuss the Unit Circle. 

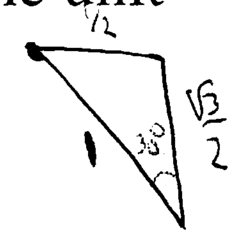
The Trigonometric Functions

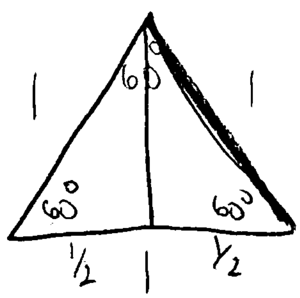
> Let t be a real number and let (x, y) be the point on the unit circle corresponding to t.
>
> $\sin t = y \qquad\qquad \csc t = \dfrac{1}{y} \quad (y \neq 0)$
>
> $\cos t = x \qquad\qquad \sec t = \dfrac{1}{x} \quad (x \neq 0)$
>
> $\tan t = \dfrac{y}{x} \quad (x \neq 0) \qquad \cot t = \dfrac{x}{y} \quad (y \neq 0)$

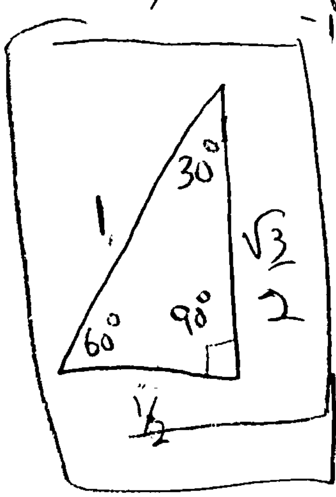
The Unit Circle

Find the the six trig values using a point on the unit circle: Let $P = \left(-\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$







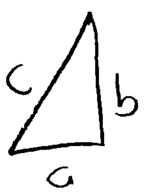


$1^2 = \left(\frac{1}{2}\right)^2 + x^2$

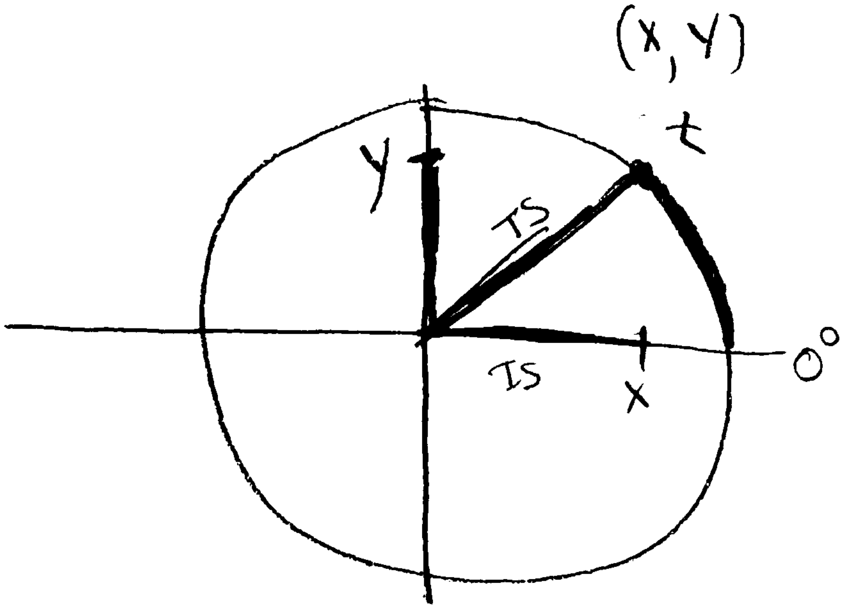
$1 = \frac{1}{4} + x^2$

$\frac{4}{4} - \frac{1}{4} = x^2$

$\sqrt{\frac{3}{4}} = \sqrt{x^2} \qquad x = \frac{\sqrt{3}}{2}$



$a^2 + b^2 = c^2$


(x, y)
t
y
TS
TS
x
0°

$\sin t = \frac{\sqrt{3}}{2}$

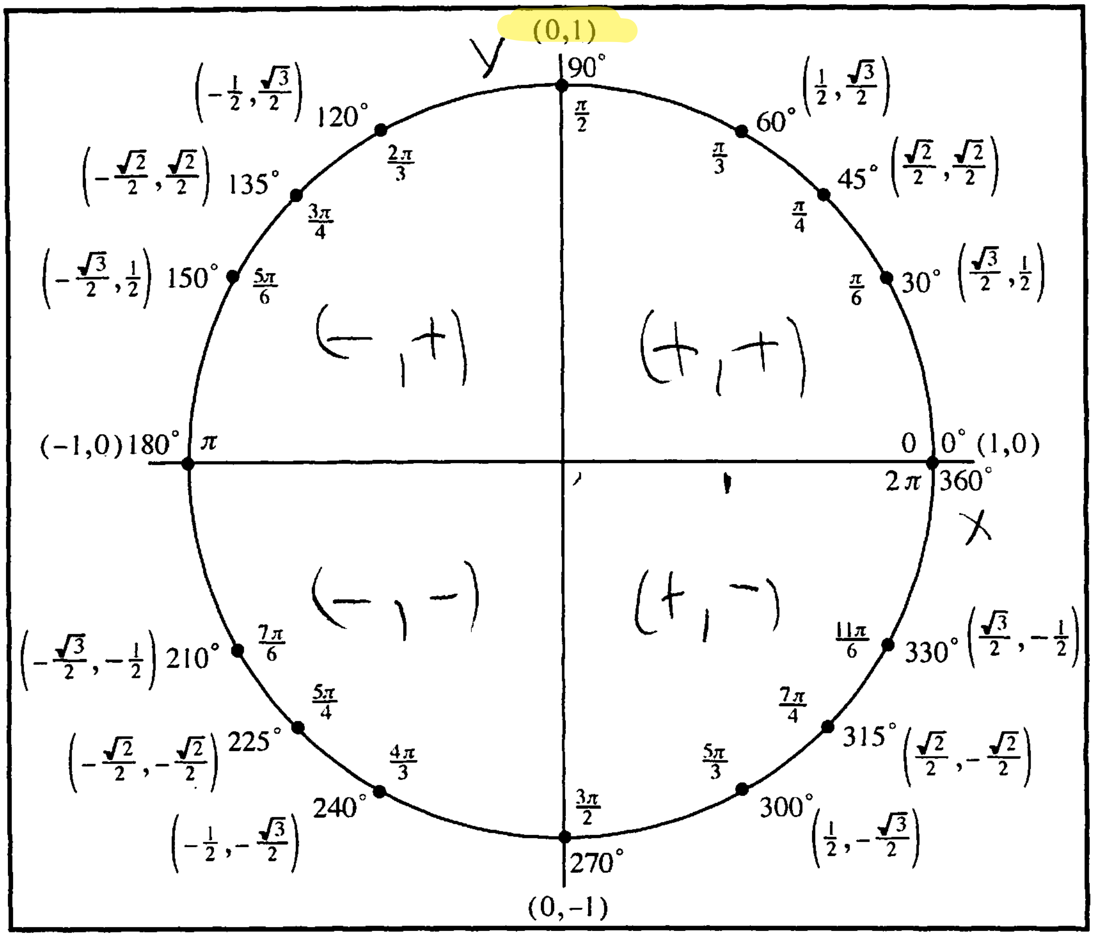
$\cos t = -\frac{1}{2}$

$\tan t = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3}$

$\csc t = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

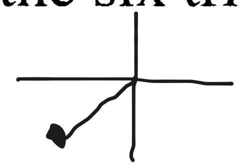
$\sec t = -\frac{2}{1} = -2$

$\cot t = \frac{1}{-\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$



$$(x,y) = (\cos\theta, \sin\theta)$$

Example Evaluate the six trig functions at each real number. $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$



$(-1,0)$

a) $t = \frac{\pi}{2} = 90°$ b) $t = \frac{5\pi}{4} = 225°$ c) $t = \pi = 180°$

a)

$\sin \frac{\pi}{2} = 1$

$\cos \frac{\pi}{2} = 0$

$\tan \frac{\pi}{2} = \frac{1}{0}$ undefined

$\csc \frac{\pi}{2} = 1$

$\sec \frac{\pi}{2} = \frac{1}{0} =$ undefined

$\cot \frac{\pi}{2} = \frac{0}{1} = 0$

b)

$\sin \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$

$\cos \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$

$\tan \frac{5\pi}{4} = 1$

$\csc \frac{5\pi}{4} = \frac{-2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{-2\sqrt{2}}{2} = -\sqrt{2}$

$\sec \frac{5\pi}{4} = \frac{-2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{-\cancel{2}\sqrt{2}}{\cancel{2}} = -\sqrt{2}$

$\cot \frac{5\pi}{4} = 1$

c)

$\sin \pi = 0$

$\cos \pi = -1$

$\tan \pi = \frac{0}{-1} = 0$

$\csc \pi = \frac{1}{0} =$ und

$\sec \pi = -1$

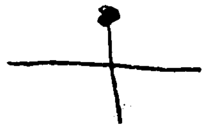
$\cot \pi = \frac{-1}{0}$ un

Examples: Find a) $\tan\frac{\pi}{4} - \sin\frac{3\pi}{2}$

$= 1 - (-1) = 2$

Examples: Find a) $\sin 135°$ b) $\cos -\frac{7\pi}{2}$ $\cdot \frac{180}{\pi}$ 90

$(-,+)$



0°

-630

a) $+\frac{\sqrt{2}}{2}$

b) $\cos -\frac{7\pi}{2}$

$\cos \frac{\pi}{2} = 0$

$\frac{360}{270}$

Discuss using a calculator.

Find $\cos 48°$ $\csc 21°$

$\frac{1}{\sin 21°}$

Definitions of Trigonometric Functions of Any Angle

Let θ be an angle in standard position with (x, y) a point on the <u>terminal</u> side of θ and

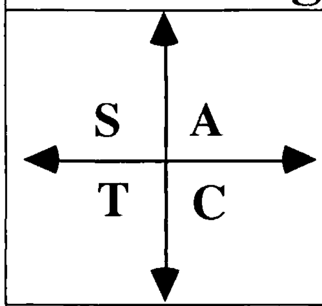
$$r = \sqrt{x^2 + y^2} \neq 0.$$

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}, \quad (x \neq 0)$$

$$\csc\theta = \frac{r}{y}, (y \neq 0) \qquad \sec\theta = \frac{r}{x}, \quad (x \neq 0)$$

$$\cot\theta = \frac{x}{y}, \quad (y \neq 0)$$

Signs of Trigonometric Functions



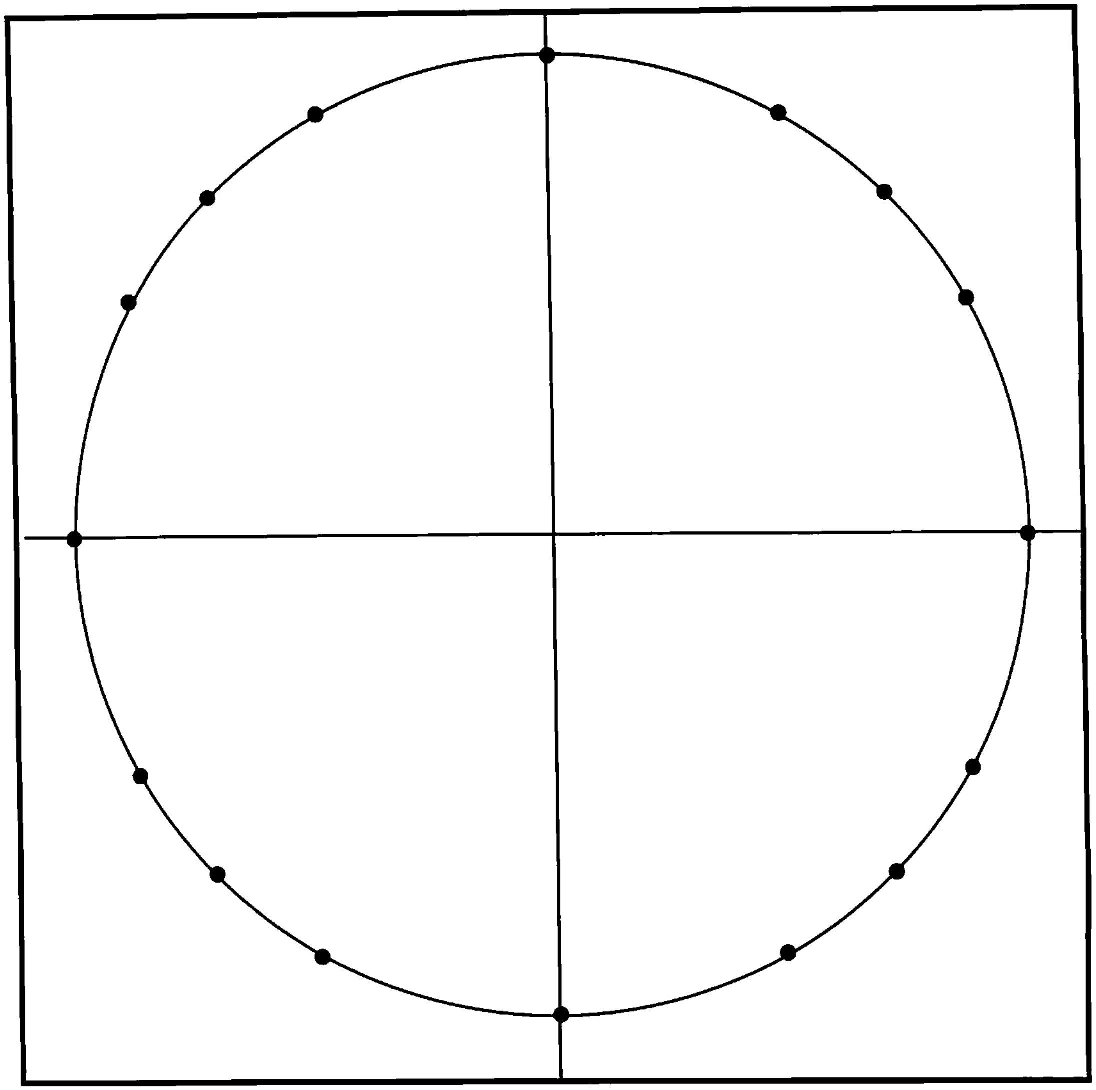
"<u>A</u>ll <u>S</u>tudents <u>T</u>ake <u>C</u>alculus"

Quad I - $\sin\theta$, $\cos\theta$, $\tan\theta$ are positive
Quad II - $\sin\theta$ is positive; $\cos\theta$, $\tan\theta$ are negative
Quad III - $\tan\theta$ is positive; $\sin\theta$, $\cos\theta$ are negative
Quad IV - $\cos\theta$ is positive; $\sin\theta$, $\tan\theta$ are negative

Example 1 Let $(4,-3)$ be a point on the terminal side of θ. Find the sine, cosine, and tangent of θ.

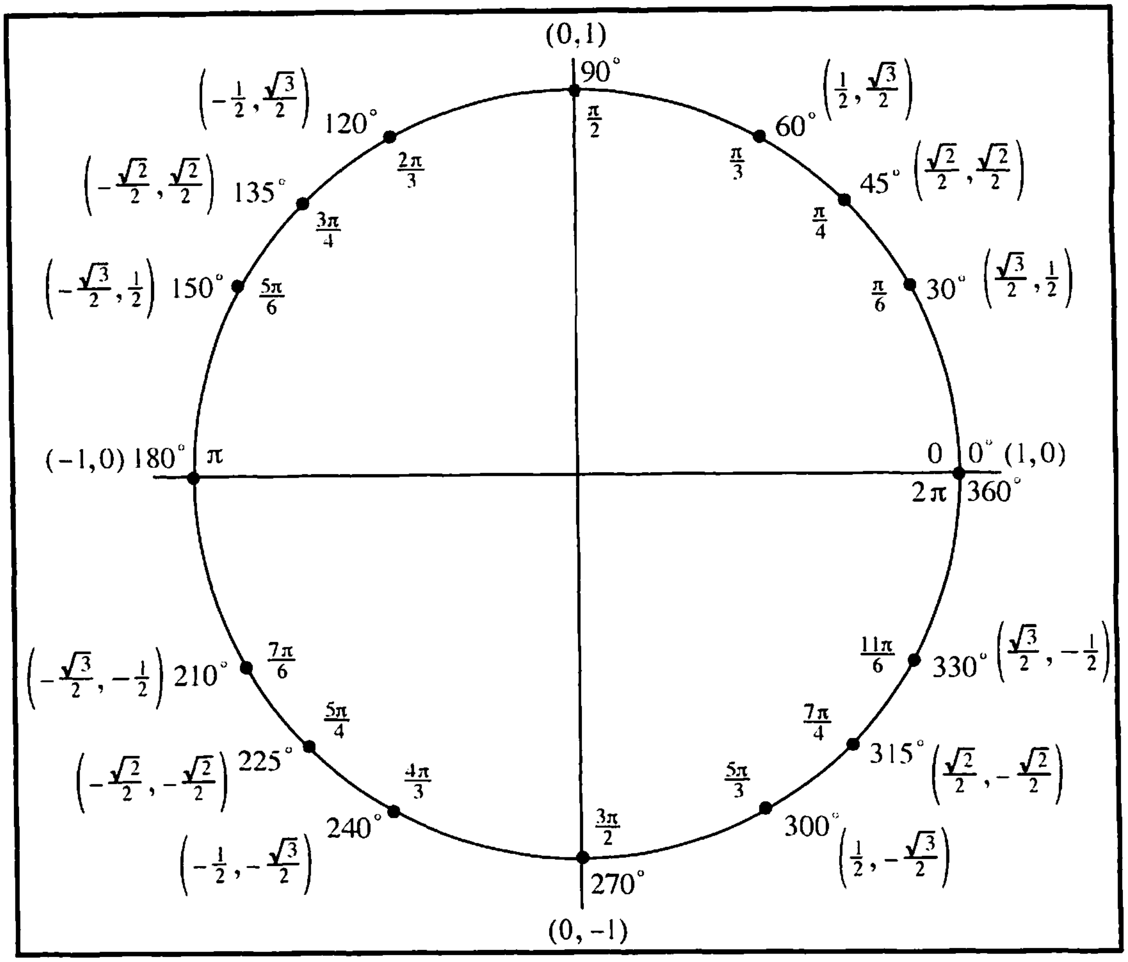
Unit Circle Worksheet

Label the unit circle below with the appropriate degrees, radians and quadrantal points.



Trigonometry Reference Sheet

The Unit Circle



$$(x, y) = (\cos\theta, \sin\theta)$$

To convert degrees to radians, multiply the number of degrees by $\frac{\pi}{180^\circ}$.

To convert radians to degrees, multiply the number of radians by $\frac{180^\circ}{\pi}$.

x degrees	x radians	$\sin x$	$\cos x$	$\tan x$
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	undefined

x degrees	x radians	**sin x**	**cos x**	**tan x**	**csc x**	**sec x**	**cot x**
0°							
30°							
45°							
60°							
90°							
120°							
135°							
150°							
180°							
210°							
225°							
240°							
270°							
300°							
315°							
330°							
360°							

§6.3 Properties of the Trigonometric Functions

Domain(Range) and Period of Sine and Cosine

Domain of the Sine and Cosine is all real numbers: $-\infty < t < \infty$	
Range of Sine	$-1 \le y \le 1$ $-1 \le \sin t \le 1$
Range of Cosine	$-1 \le x \le 1$ $-1 \le \cos t \le 1$

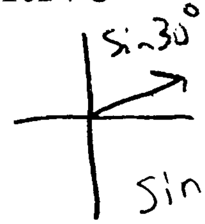
Definition of a periodic function

A function f is **periodic** if there exists a positive real number c such that

$$f(t + c) = f(t)$$

Coterminal

$\sin 30^\circ = \frac{1}{2}$



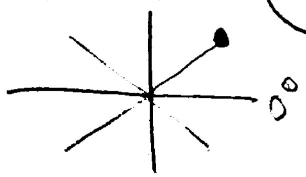
$\sin 390^\circ = \frac{1}{2}$

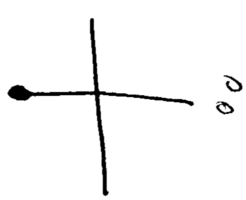
for all t in the domain of f. The smallest number c for which f is periodic is called the **period** of f.

Examples: Find

a) $\sin \frac{17\pi}{4} \cdot \frac{180}{\pi}$ b) $\cos 5\pi$ c) $\tan \frac{5\pi}{4}$

$\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$

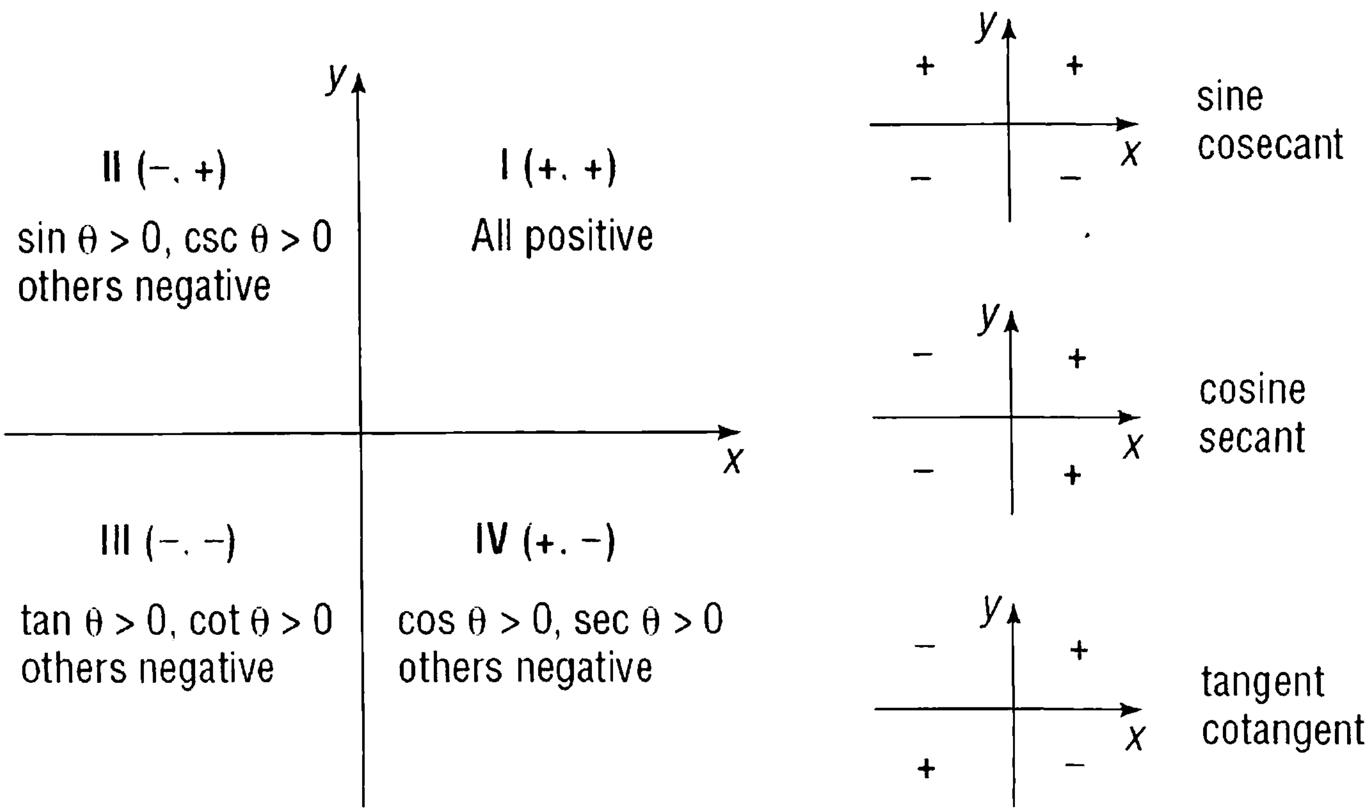




$\cos \pi = -1$

$+1$

Determine the signs of the Trig Functions in a Given Quadrant

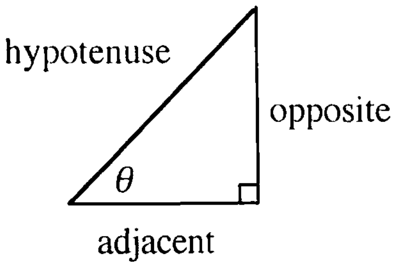


If sin ø < 0 and cos ø < 0, name the quadrant in which the angle lies.

III

Find the Values of the Trig Functions Using Fundamental Identities

Right Triangle Trigonometry



Right Triangle Definitions of Trigonometric Functions

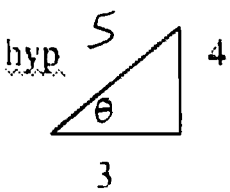
sohcahtoa

Let θ be an acute angle of a right triangle. Then:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

Example Evaluate the six trig functions for this triangle:



$3^2 + 4^2 = x^2$

$9 + 16 = x^2$

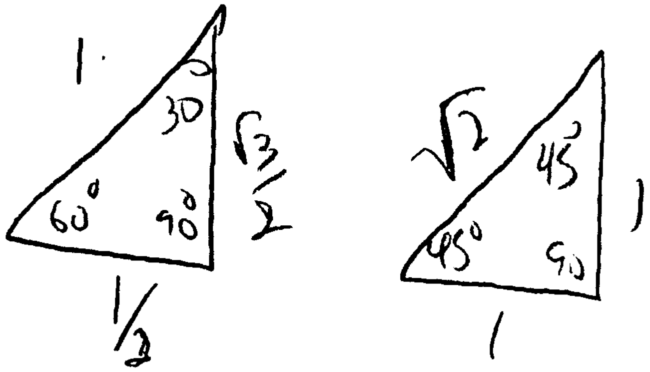
$x^2 = 25$

$x = 5$

$\sin\theta = \frac{4}{5}$ $\quad \csc\theta = \frac{5}{4}$

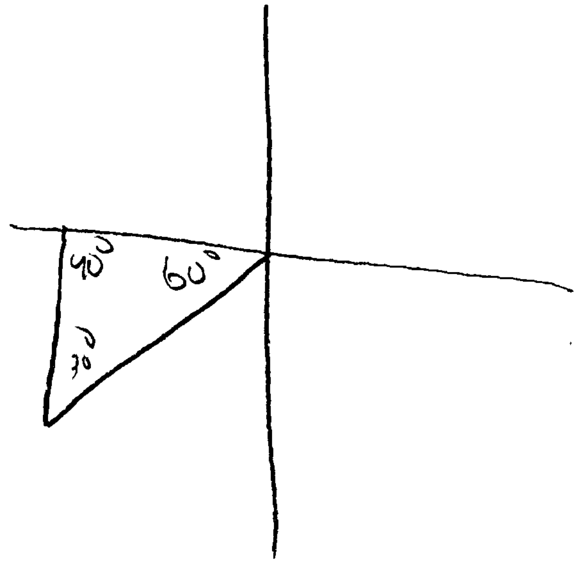
$\cos\theta = \frac{3}{5}$ $\quad \sec\theta = \frac{5}{3}$

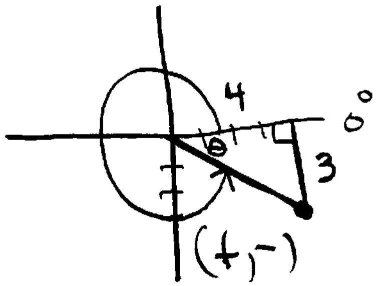
$\tan\theta = \frac{4}{3}$ $\quad \cot\theta = \frac{3}{4}$



Sohcah toa

A S T C





$r = \sqrt{x^2 + y^2}$

$= \sqrt{(4)^2 + (-3)^2}$

$= \sqrt{16 + 9}$

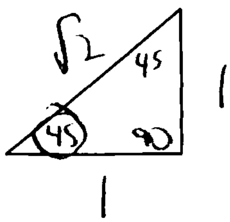
$= \sqrt{25} = 5$

$\sin = \frac{y}{r} = \frac{-3}{5}$

$\cos = \frac{x}{r} = \frac{4}{5}$

$\tan = \frac{y}{x} = \frac{-3}{4}$

Example : Find the values of $\sin 45^\circ$, $\cos 45^\circ$, and $\tan 45^\circ$ using right triangle trigonometry.

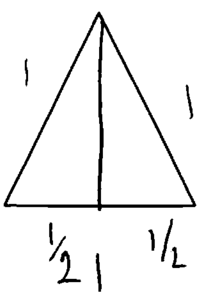


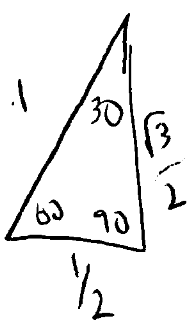
$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45^\circ = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45^\circ = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

Example : Find the values of $\sin 30^\circ$, $\cos 30^\circ$, $\sin 60^\circ$, and $\cos 60^\circ$ using right triangle trigonometry.





$$\sin 30^\circ = \frac{1}{2} \quad \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\sin 60 = \frac{\sqrt{3}}{2} \quad \cos 60^\circ = \frac{1}{2}$$

$$\frac{\sin^2\theta}{\cos^2\theta} + \cancel{\frac{\cos^2\theta}{\cos^2\theta}} = \frac{1}{\cos^2\theta}$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\cancel{\frac{\sin^2\theta}{\sin^2\theta}} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$1 + \cot^2\theta = \csc^2\theta$$

Trigonometric Identities

Reciprocal Identities		
$\sin\theta = \dfrac{1}{\csc\theta}$	$\cos\theta = \dfrac{1}{\sec\theta}$	$\tan\theta = \dfrac{1}{\cot\theta}$
$\csc\theta = \dfrac{1}{\sin\theta}$	$\sec\theta = \dfrac{1}{\cos\theta}$	$\cot\theta = \dfrac{1}{\tan\theta}$

Quotient or Ratio Identities	
$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Pythagorean Identities	
$\sin^2\theta + \cos^2\theta = 1$	$\tan^2\theta + 1 = \sec^2\theta$
$1 + \cot^2\theta = \csc^2\theta$	

Example: Find the exact values of the remaining four trig functions of θ using identities.

$$\sin\theta = \frac{\sqrt{5}}{5} \quad \text{and} \quad \cos\theta = \frac{2\sqrt{5}}{5}$$

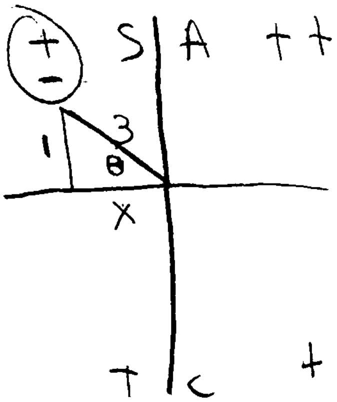
$\csc\theta = \frac{5}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{5\sqrt{5}}{5} = \sqrt{5}$

$\sec\theta = \frac{5}{2\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{5\sqrt{5}}{10} = \frac{\sqrt{5}}{2}$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{\sqrt{5}}{5}}{\frac{2\sqrt{5}}{5}} = \frac{\cancel{\sqrt{5}}}{\cancel{5}} \cdot \frac{\cancel{5}}{2\cancel{\sqrt{5}}} = \frac{1}{2}$

$\cot\theta = 2$

Example : Given that $\sin\theta = \frac{1}{3}$ and $\cos\theta < 0$, find the exact value of each of the remaining five trig functions of θ.



$\sin\theta = \frac{1}{3}$

$\cos\theta = -\frac{2\sqrt{2}}{3}$

$\tan\theta = \frac{1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{2}}{4}$

$\csc\theta = 3$

$\sec\theta = -\frac{3}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = -\frac{3\sqrt{2}}{4}$

$\cot\theta = -\frac{4}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$

$= -\frac{4\sqrt{2}}{2}$

$= -2\sqrt{2}$

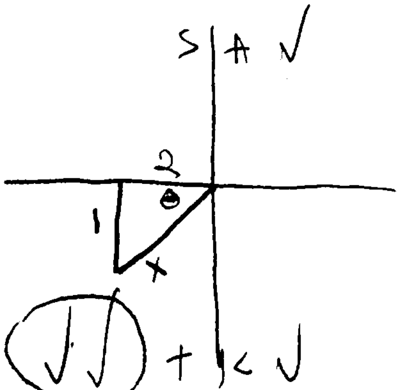
$3^2 = 1^2 + x^2$

$9 = 1 + x^2$

$\sqrt{x^2} = \sqrt{8}$

$x = 2\sqrt{2}$

Example : Given that $\tan\theta = \frac{1}{2}$ and $\sin\theta < 0$, find the exact value of each of the remaining five trig functions of θ.



$\cot\theta = \frac{2}{1}$

$\sin\theta = \frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$

$\cos\theta = \frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$

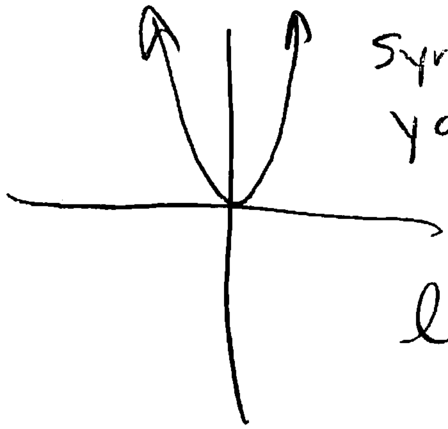
$\sec\theta = -\frac{5}{2\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = -\frac{5\sqrt{5}}{10} = -\frac{\sqrt{5}}{2}$

$\csc\theta = -\frac{\sqrt{5}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = -\frac{5\sqrt{5}}{5} = -\sqrt{5}$

$x^2 = 1^2 + 2^2$

$x^2 = 5$

$x = \sqrt{5}$



sym. $y = x^2$
y axis

even

odd origin

Even and Odd Trigonometric Functions

The cosine and secant functions are **even**.

$\cos(-t) = \cos t \qquad \sec(-t) = \sec t$

The sine, cosecant, tangent, and cotangent functions are **odd**.

$\sin(-t) = -\sin(t) \qquad \csc(-t) = -\csc(t)$

$\tan(-t) = -\tan(t) \qquad \cot(-t) = -\cot(t)$

Find the exact value of:

a) $\sin(-45°)$

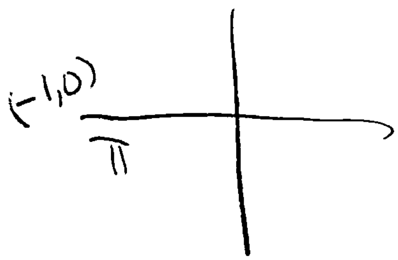
$= -\sin(45°)$

$= -\frac{\sqrt{2}}{2}$

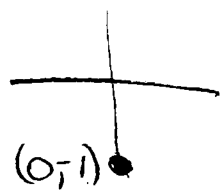
b) $\cos(-\pi)$

$= \cos(\pi)$

$= -1$



c) $\cot\left(-\frac{3\pi}{2}\right)$



$-\cot\left(\frac{3\pi}{2}\right)$

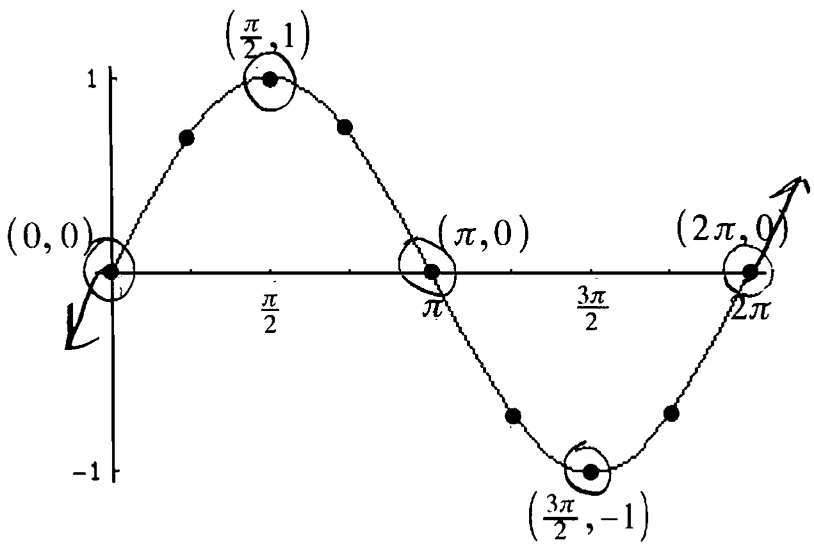
$= 0$

§6.4 Graphs of Sine and Cosine Functions

Graph of $y = \sin x$

APTEV

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
y = sin x	0		1		0		-1		0



- since the domain of $y = \sin x$ is all real numbers, the graph repeats infinitely to the left and the right

- one <u>period</u> (or <u>cycle</u>) of the graph is on $[0,2\pi]$

Graphing trigonometric functions on TI calculator

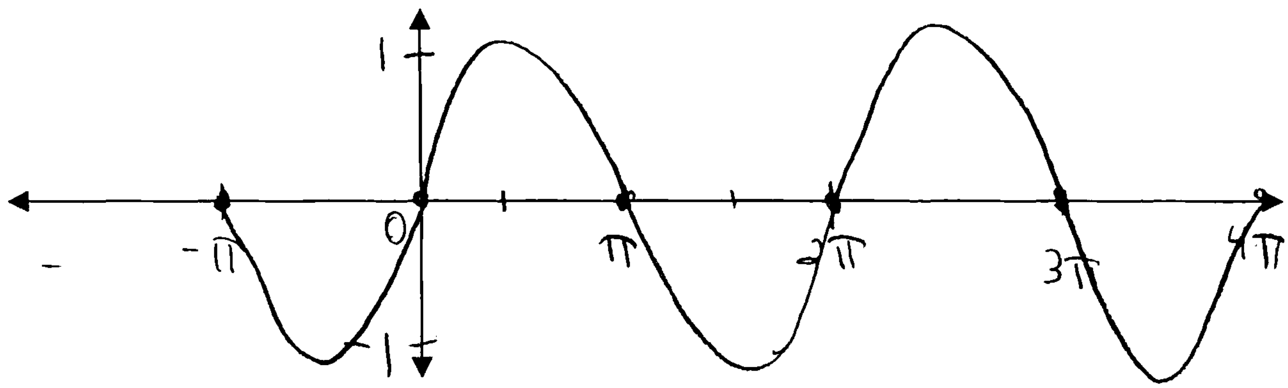
$\boxed{\text{MODE}}$ all choices on left should be highlighted, radians

$\boxed{\text{WINDOW}}$

xmin	-2π	
xmax	2π	
xscl	$\pi/2$	(tick marks)
ymin	-2	
ymax	2	
yscl	1	

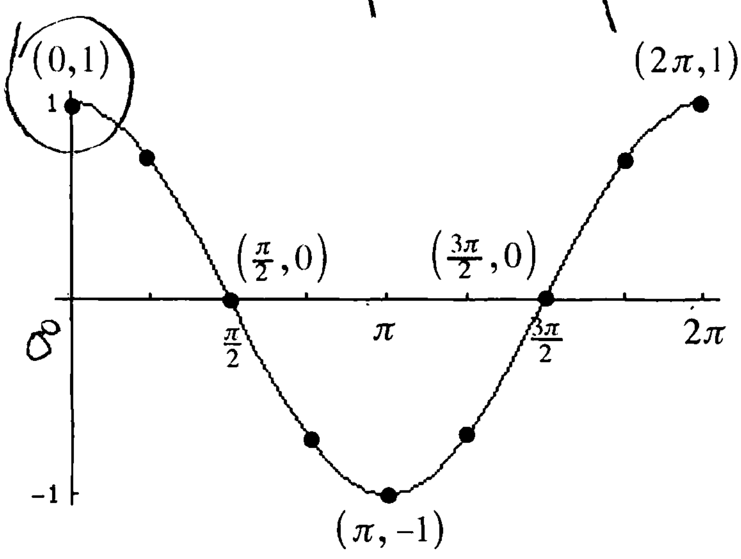
Example: Graph y = sin x on your calculator. Draw the axes and label properly.

Example: Sketch the graph of y = 2 sin x on the interval $[-\pi, 4\pi]$. Remember key points.



Graph of y = cos x

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
y = cos x	1		0		-1		0		1



- since the domain of y = cos x is all real numbers, the graph repeats infinitely to the left and the right

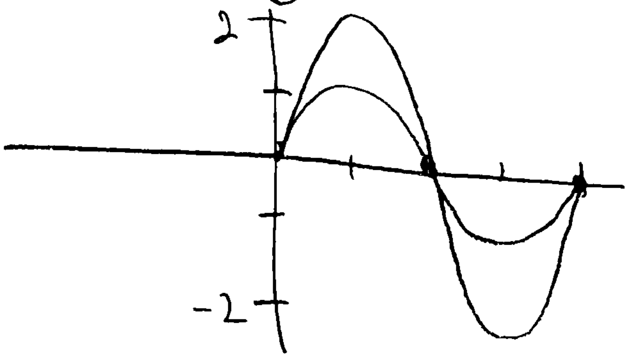
- one <u>period</u> (or <u>cycle</u>) of the graph is on $[0, 2\pi]$

Amplitude

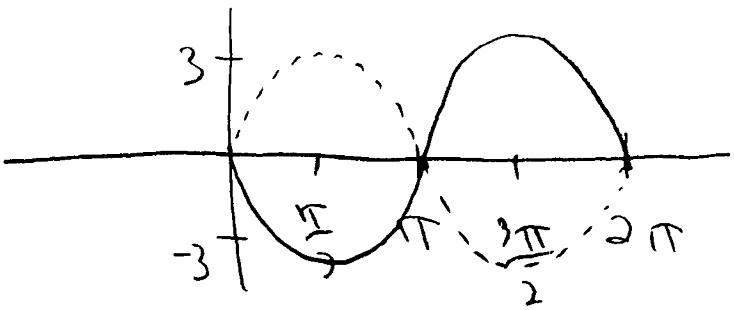
Compare the graph $y = \sin x$ to each of the following: <u>(Vertical Shrinking and Stretching)</u>

$y = 1\sin x$

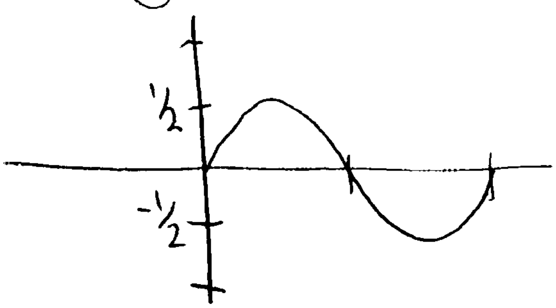
Ex 1. $y = 2\sin x$



2. $y = -3\sin x$



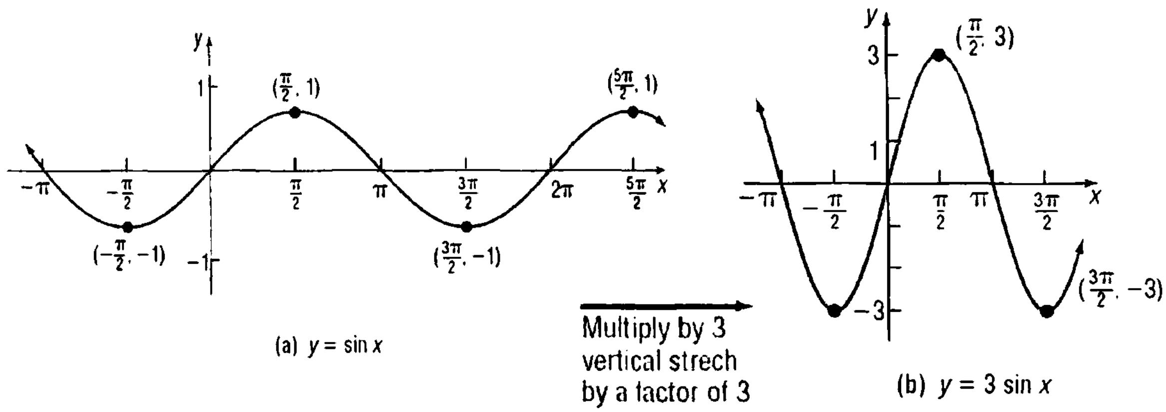
3. $y = \frac{1}{2}\sin x$



EXAMPLE

Graphing Functions of the Form $y = A\sin(\omega x)$ Using Transformations

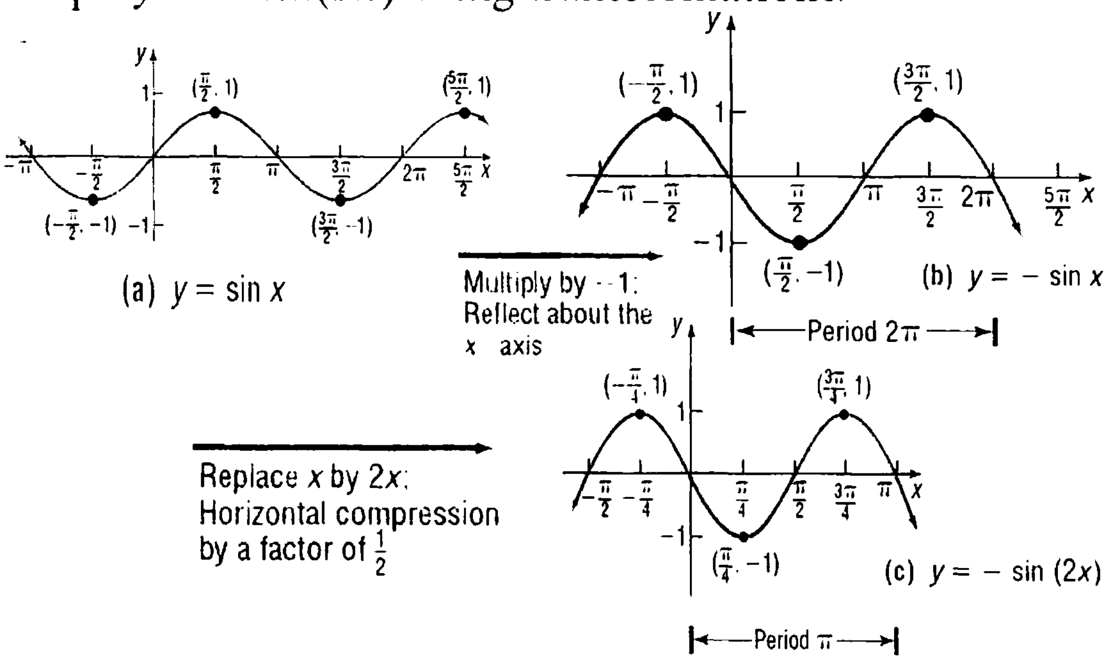
Graph $y = 3\sin x$ using transformations.



EXAMPLE

Graphing Functions of the Form $y = A\sin(\omega x)$ Using Transformations

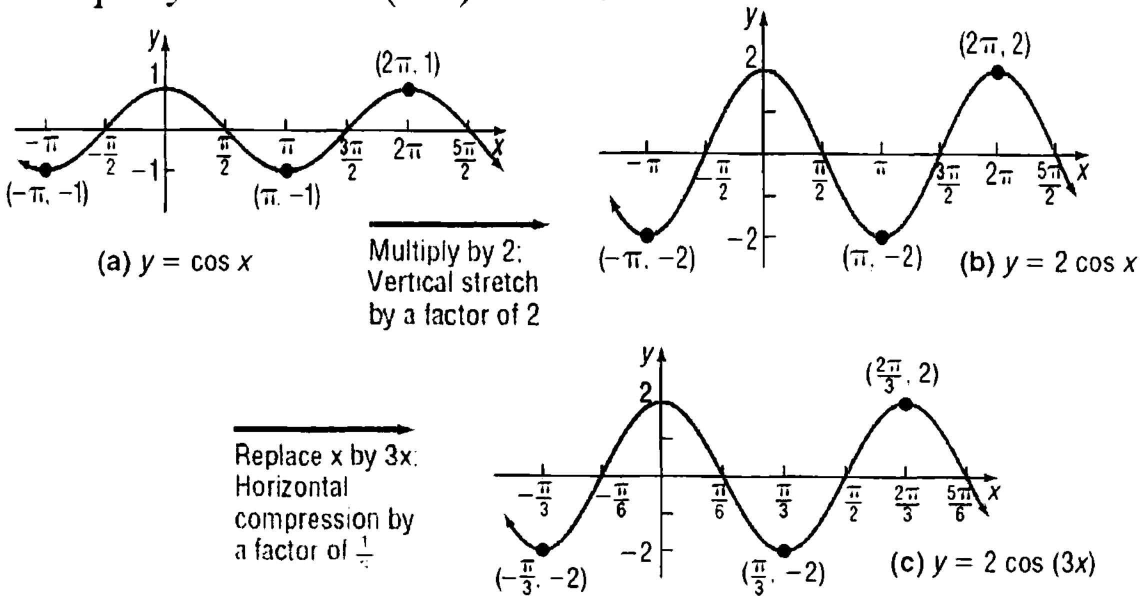
Graph $y = -\sin(2x)$ using transformations.



EXAMPLE

Graphing Functions of the Form $y = A\cos(\omega x)$ Using Transformations

Graph $y = 2\cos(3x)$ using transformations.



APTEV

Formulas for General Form $y = a\sin(bx - c) + d$ **and** $y = a\cos(bx - c) + d$

amplitude = $|a|$

period (of sine and cosine) = $\dfrac{2\pi}{b}$

tick marks = $\dfrac{\text{period}}{4}$

endpoints   Solve: $bx - c = 0$ $bx - c = 2\pi$

vertical shift = d

Example: Horizontal Translation

Sketch the graph of $y = \frac{1}{2}\sin\left(x - \frac{\pi}{3}\right) + 0$

$y = a\sin(bx - c) + d$

Example: Horizontal Translation

Sketch the graph of $y = -3\cos(2\pi x + 4\pi)$

Example: Vertical Translation

Sketch the graph of $y = 2 + 3\cos(2x)$

$y = 3\cos(2x) + 2$

Example: $y=\frac{1}{2}\sin\left(x-\frac{\pi}{3}\right)$ (Remember APTEV)

Formulas for General Form $y=a\sin(bx-c)+d$ and $y=a\cos(bx-c)+d$

amplitude = $\lvert a\rvert=\left\lvert\frac{1}{2}\right\rvert=\frac{1}{2}$	tick mark calculations: (Key points)
period (of sine and cosine) = $\frac{2\pi}{b}=\frac{2\pi}{1}=2\pi$	(1) $\frac{\pi}{3}$ (2) $\frac{2}{2}\cdot\frac{\pi}{3}+\frac{\pi}{2}\cdot\frac{3}{3}=\frac{5\pi}{6}$ (3) $\frac{5\pi}{6}+\frac{\pi}{2}=\frac{8\pi}{6}=\frac{4\pi}{3}$ (4) $\frac{4\pi}{3}+\frac{\pi}{2}=\frac{11\pi}{6}$
tick marks = $\frac{\text{period}}{4}=\frac{2\pi}{4}=\frac{\pi}{2}$	(5) $\frac{11\pi}{6}+\frac{\pi}{2}\cdot\frac{3}{3}=\frac{7\pi}{3}$

endpoints Solve:

$bx-c=0$

$x-\frac{\pi}{3}=0$

$x=\frac{\pi}{3}$

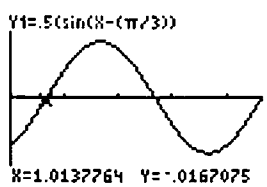
(starts)

$bx-c=2\pi$

$x-\frac{\pi}{3}=2\pi$

$x=\frac{\pi}{3}+\frac{6\pi}{3}=\frac{7\pi}{3}$

(ends)



vertical shift = d = none

$y = a\cos(bx - c) + d$

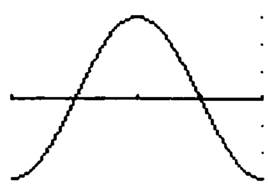
Example : $y = -3\cos(2\pi x + 4\pi)$ (Remember APTEV)

Formulas for General Form $y = a\sin(bx - c) + d$ and $y = a\cos(bx - c) + d$

amplitude = $\lvert a\rvert = \lvert -3\rvert = 3$	tick mark calculations:
period (of sine and cosine) = $\frac{2\pi}{b} = \frac{2\pi}{2\pi} = 1$	(1) -2 (2) $\frac{4}{4}\cdot\frac{-2}{1} + \frac{1}{4} = \frac{-7}{4}$ (3) $\frac{-7}{4} + \frac{1}{4} = \frac{-6}{4} = \frac{-3}{2}$
tick marks = $\frac{\text{period}}{4} = \frac{1}{4}$	(4) $\frac{-3}{2} + \frac{1}{4} = \frac{-5}{4}$ (5) $\frac{-5}{4} + \frac{1}{4} = -1$

endpoints Solve:

$bx - c = 0$	$bx - c = 2\pi$
$2\pi x + 4\pi = 0$	$2\pi x + 4\pi = 2\pi$
$2\pi x = -4\pi$	$2\pi x = -2\pi$
$x = -2$	$x = -1$
(starts)	(ends)



vertical shift = d = none

Example: $y = 3\cos(2x) + 2$ (Remember APTEV)

Formulas for General Form $y = a\sin(bx - c) + d$ and $y = a\cos(bx - c) + d$

amplitude = $|a| = |3| = 3$

period (of sine and cosine) =

$$\frac{2\pi}{b} = \frac{2\pi}{2} = \pi$$

tick marks = $\dfrac{\text{period}}{4} = \dfrac{\pi}{4}$

tick mark calculations:

(1) 0

(2) $0 + \dfrac{\pi}{4} = \dfrac{\pi}{4}$

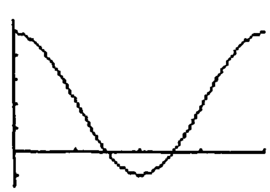
(3) $\dfrac{\pi}{4} + \dfrac{\pi}{4} = \dfrac{\pi}{2}$ (4)

4) $\dfrac{\pi}{2} + \dfrac{\pi}{4} = \dfrac{3\pi}{4}$

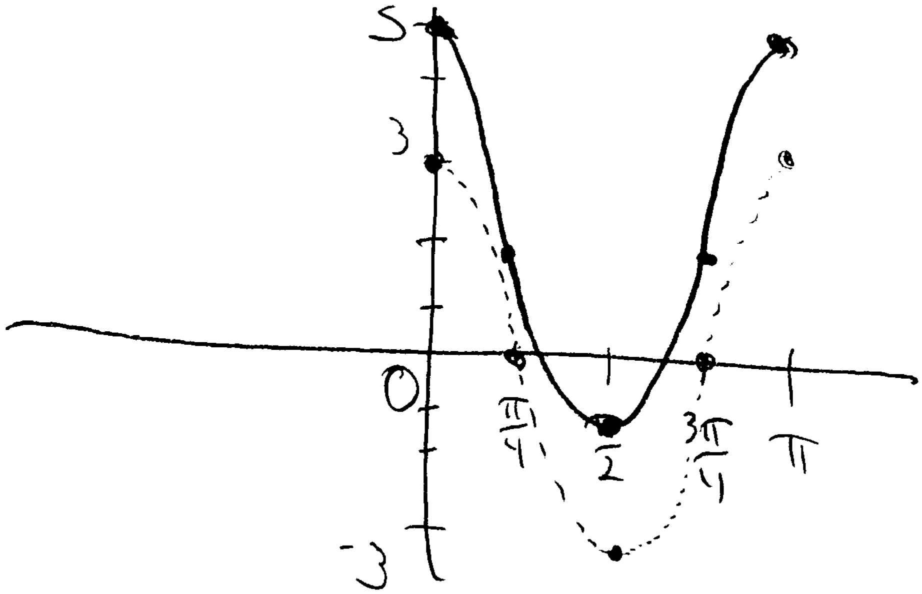
(5) $\dfrac{3\pi}{4} + \dfrac{\pi}{4} = \dfrac{4\pi}{4} = \pi$

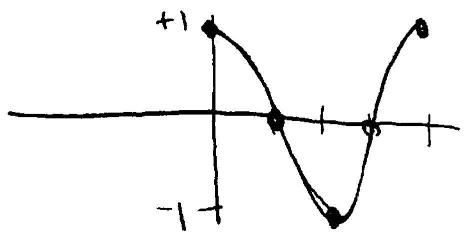
endpoints Solve:

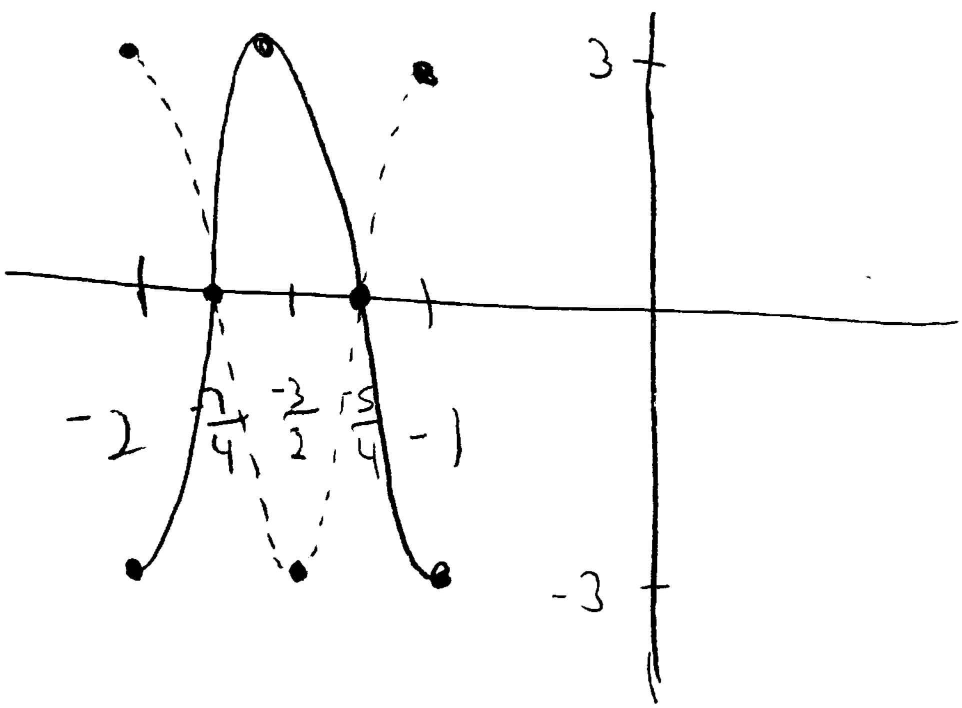
$bx - c = 0$	$bx - c = 2\pi$
$2x - 0 = 0$	$2x - 0 = 2\pi$
$x = 0$	$x = \pi$
(starts)	(ends)

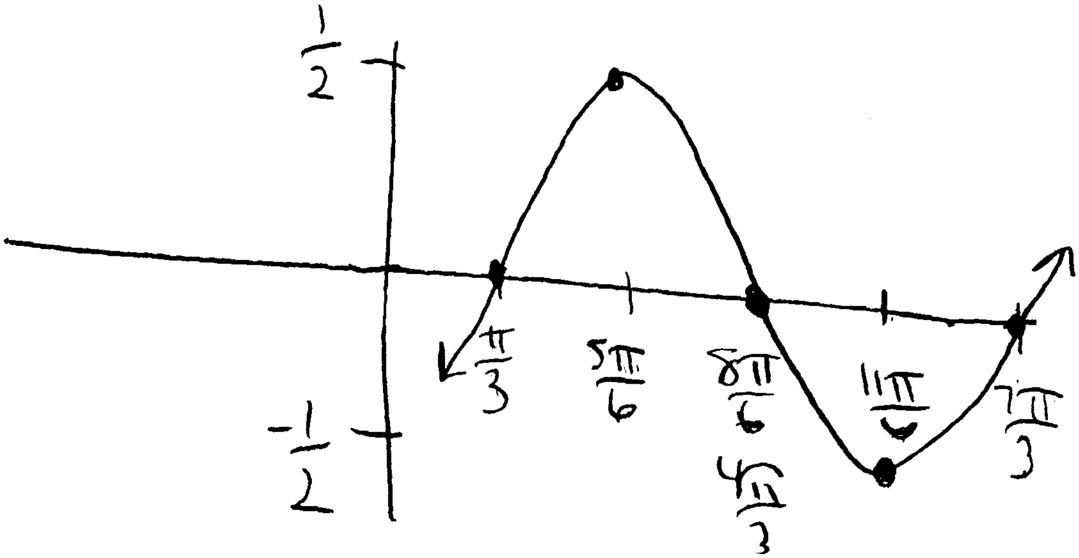


vertical shift = d = 2

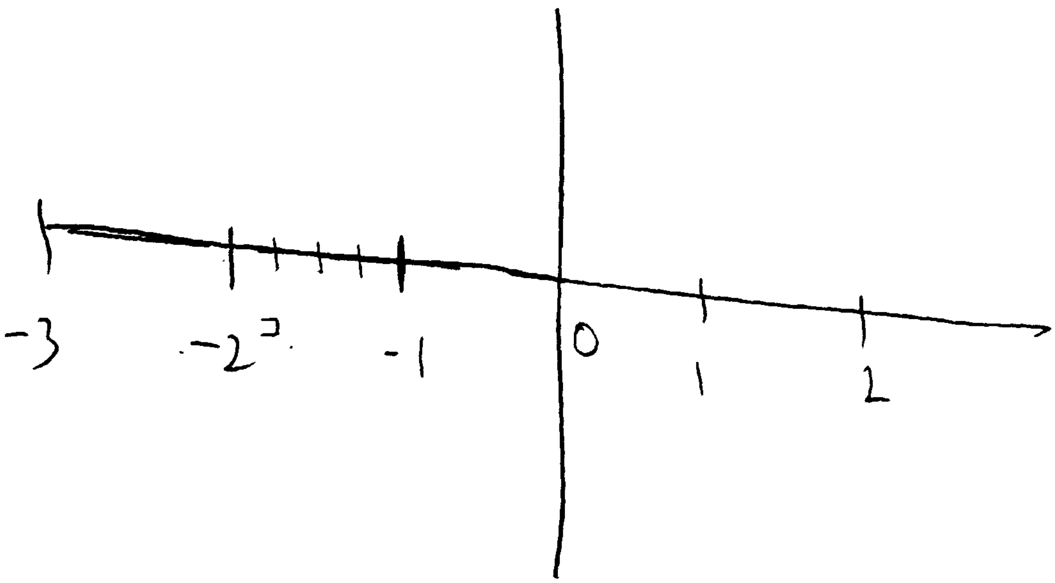

5
3
-3
0
π/4
π/2
3π/4
π


+1
-1


3
-3
-2
-7/4
-3/2
-5/4
-1



1/2
-1/2
π/3
5π/6
8π/6
4π/3
11π/6
7π/3



-3
-2
-1
0
1
2

Requirements for a Correct Trigonometric Graph

In order to receive full credit for a graph, you must do all of the following.

1.) Label your axes.

2.) Show at least one period.

3.) Label five ordered pairs or asymptotes (as appropriate).

Generalized Formulas for Graphs

$y = a\ \sin(bx - c) + d$	$y = a\ \csc(bx - c) + d$
$y = a\ \cos(bx - c) + d$	$y = a\ \sec(bx - c) + d$
$y = a\ \tan(bx - c) + d$	$y = a\ \cot(bx - c) + d$

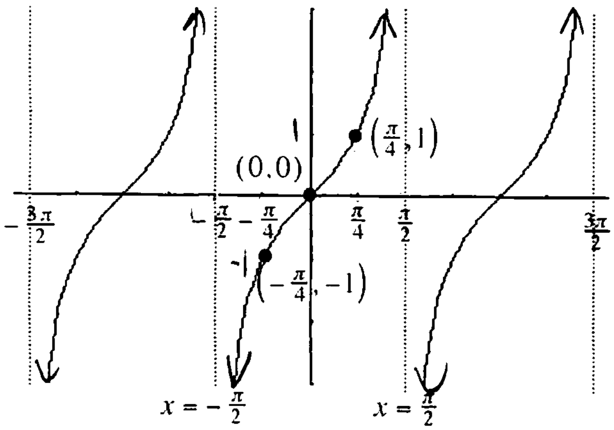
amplitude = $\lvert a\rvert$	amplitude only exists for sine and cosine changes the y axis on all graphs if a < 0, reflection about x-axis
period	$\frac{2\pi}{b}$ for sin, cos, csc, sec $\frac{\pi}{b}$ for tangent and cotangent
tick marks	$\frac{period}{4}$ for all trig functions
endpoints	$bx - c = 0$ and $bx - c = 2\pi$ for sin, cos, csc, sec graphs $bx - c = \frac{-\pi}{2}$ and $bx - c = \frac{\pi}{2}$ for tangent graph $bx - c = 0$ and $bx - c = \pi$ for cotangent graph
vertical shift	d for all trig functions

§6.5 Graphs of the Other Trigonometric Functions

Graph of y = tan x $= \frac{\sin x}{\cos x}$

x	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
tan x	undefined	-1	0	1	undefined

$\frac{-1}{0}$



since the domain of y = tan x is all real numbers except $\frac{(2n+1)\pi}{2}$, the graph repeats infinitely to the left and the right

one <u>period</u> (or <u>cycle</u>) of the graph is on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Example 1: Graph a) $y = \tan\frac{x}{2}$ b) $y = -3\tan 2x$

$y = a\cos(bx - c) + d$

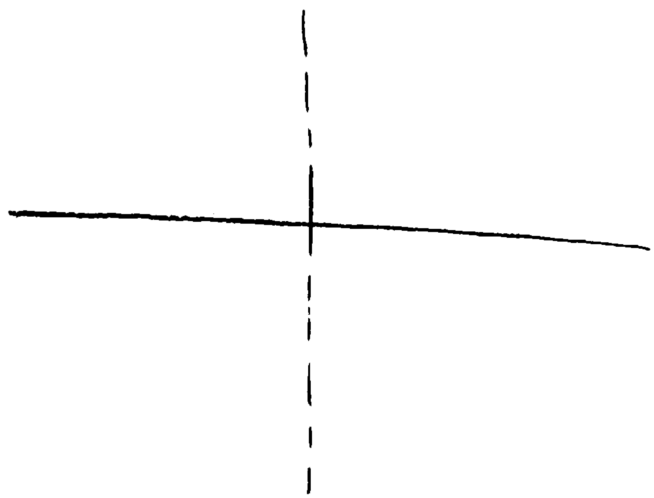
$y = a\sin(bx - c) + d$

$y = a\tan(bx - c) + d$

$$y = \frac{1}{x}$$

$x = 0$?

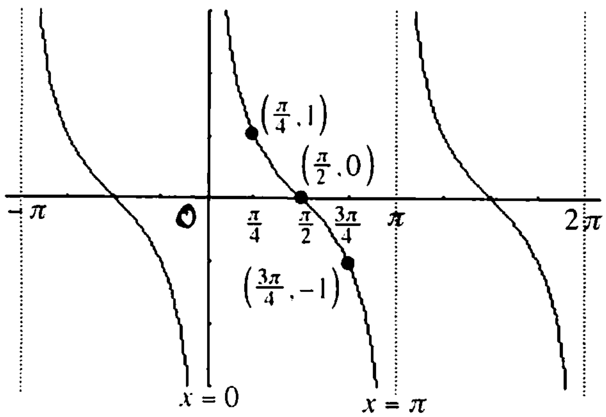
$y =$ undefined



Graph of y = cot x $= \dfrac{\cos x}{\sin x}$

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
cot x	undefined	1	0	-1	undefined

$1/0$ $\qquad\qquad\qquad$ $0/1$



since the domain of y = cot x is all real numbers except $n\pi$, the graph repeats infinitely to the left and the right

one <u>period</u> (or <u>cycle</u>) of the graph is on $[0,\pi]$

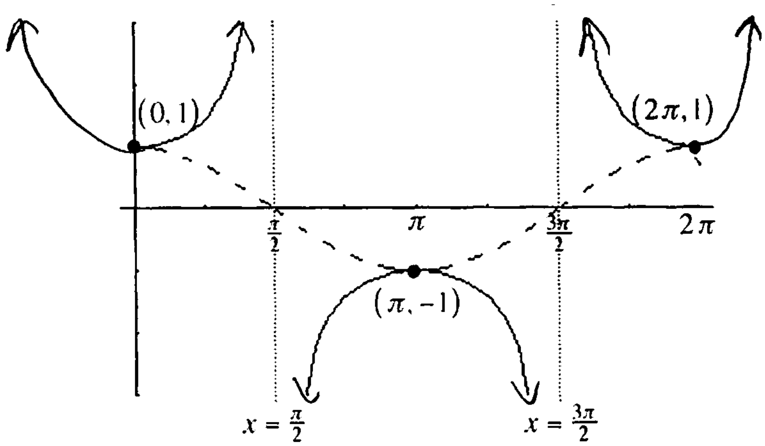
Example 2: Graph $y = 2\cot\dfrac{x}{3}$

$y = a\cot(bx - c) + d$

$y = 2\cos\left(\frac{1}{3}x - 0\right) + 0$

Graph of y = sec (x) $= \frac{1}{\cos x}$

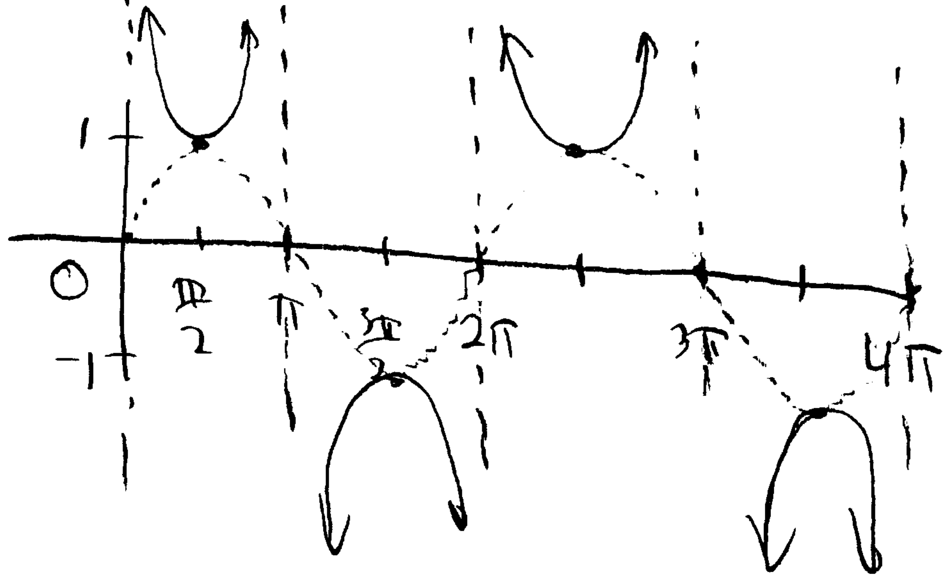
x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y = sec x	1	undefined	−1	undefined	1



since the domain of y = sec x is all real numbers except $\frac{(2n+1)\pi}{2}$, the graph repeats infinitely to the left and the right

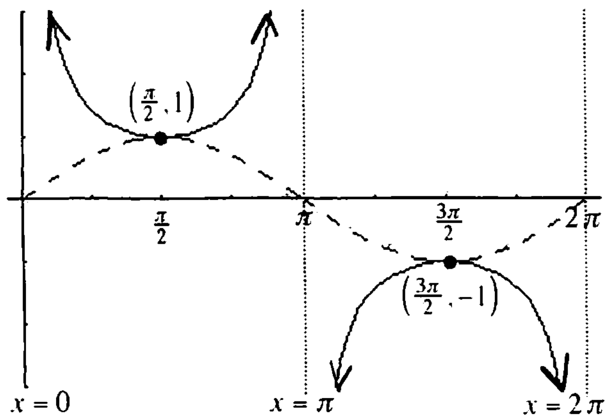
one period (or cycle) of the graph is on $[0, 2\pi]$

Example 3: Graph a) $y = 2\csc\left(x + \frac{\pi}{4}\right)$ b) $y = \sec(2x)$


1
-1
0
π/2
π
3π/2
2π
3π
4π

Graph of y = csc (x) $= \frac{1}{\sin x}$

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y = csc x	undefined	1	undefined	-1	undefined



since the domain of $y = \csc x$ is all real numbers except $n\pi$, the graph repeats infinitely to the left and the right

one period (or cycle) of the graph is on $[0, 2\pi]$

Example: $y = \tan\left(\frac{x}{2}\right)$ (Remember APTEV)

$y = 1\tan\left(\frac{1}{2}x - 0\right) + 0$

Formulas for General Form $y = a\tan(bx - c) + d$

amplitude = none

period (of tan and cot) =

$\boxed{\frac{\pi}{b}} = \frac{\pi}{1/2} = 2\pi$

$\frac{\frac{\pi}{1}}{\frac{1}{2}} = \frac{\pi}{1} \cdot \frac{2}{1}$

tick marks = $\frac{\text{period}}{4} = \frac{2\pi}{4} = \frac{\pi}{2}$

tick mark calculations:

(1) $-\pi$

(2) $-\pi + \frac{\pi}{2} = \frac{-\pi}{2}$

(3) $\frac{-\pi}{2} + \frac{\pi}{2} = 0$

(4) $0 + \frac{\pi}{2} = \frac{\pi}{2}$

(5) $\frac{\pi}{2} + \frac{\pi}{2} = \pi$

endpoints Solve:

$\boxed{bx - c = \frac{-\pi}{2}}$

$\frac{x}{2} = \frac{-\pi}{2}$

$x = -\pi$

(starts)

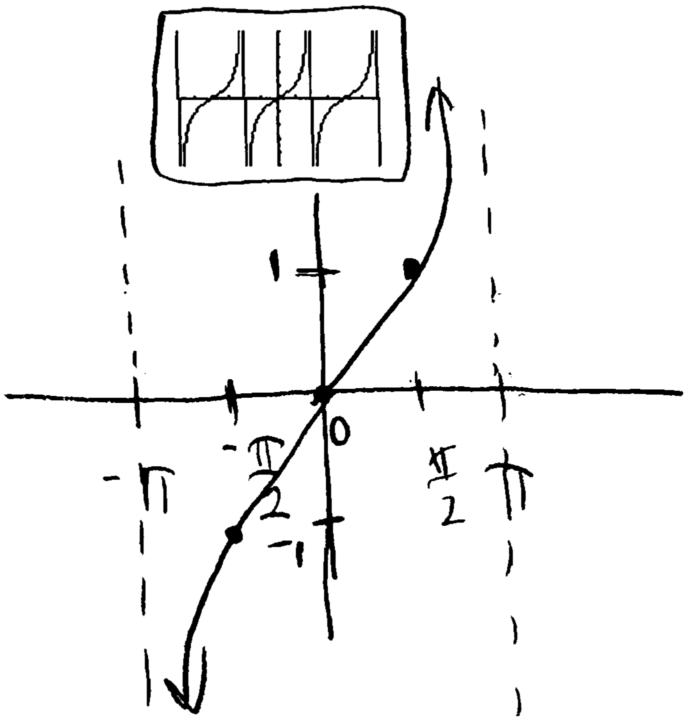
$\boxed{bx - c = \frac{\pi}{2}}$

$\frac{x}{2} = \frac{\pi}{2}$

$x = \pi$

(ends)

vertical shift = none



Example: $y = 2\cot\left(\frac{x}{3}\right)$ (Remember APTEV)

Formulas for General Form $y = a\ \cot(bx - c) + d$

amplitude = $\boxed{\text{none}}$ 2

period (of tan and cot) =

$\frac{\pi}{b} = \frac{\pi}{1/3} = 3\pi$

tick marks = $\frac{\text{period}}{4} = \boxed{\frac{3\pi}{4}}$

tick mark calculations:

(1) 0

(2) $0 + \frac{3\pi}{4} = \frac{3\pi}{4}$

(3) $\frac{3\pi}{4} + \frac{3\pi}{4} = \frac{3\pi}{2}$ $\frac{6\pi}{4}$

(4) $\frac{2}{2}\,\frac{3\pi}{2} + \frac{3\pi}{4} = \frac{9\pi}{4}$

(5) $\frac{9\pi}{4} + \frac{3\pi}{4} = 3\pi$ $\frac{12\pi}{4}$

endpoints Solve:

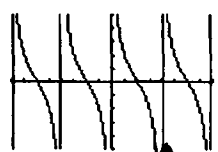
$\boxed{bx - c = 0}$ $\boxed{bx - c = \pi}$

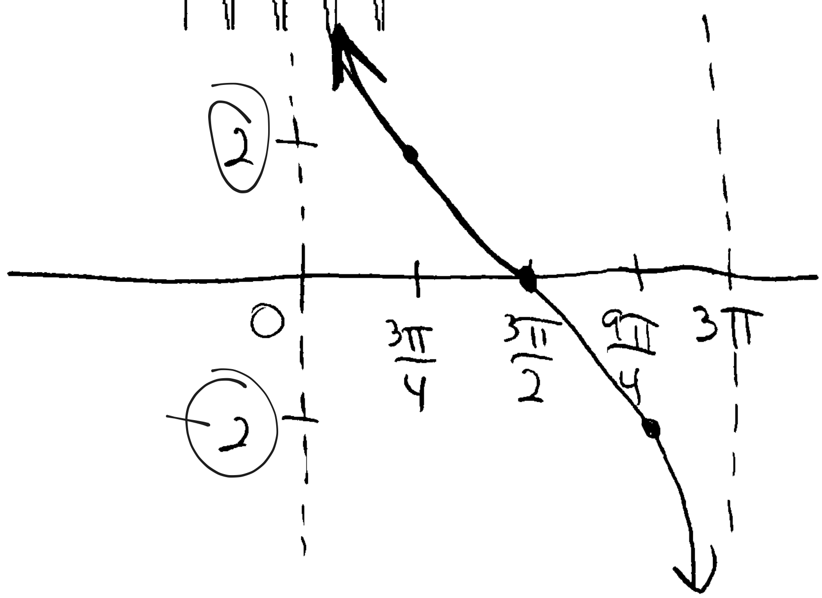
$\frac{x}{3} = 0$ $\frac{x}{3} = \pi$

$x = 0$ $x = 3\pi$

(starts) (ends)

vertical shift = none





Example: $y = 2\csc\left(x + \frac{\pi}{4}\right)$ (Remember APTEV)

Formulas for General Form $y = a\sin(bx - c) + d$ **and** $y = a\cos(bx - c) + d$

amplitude $= |a| = |2| = 2$

period (of sine and cosine) =

$\frac{2\pi}{b} = \frac{2\pi}{1} = 2\pi$

tick marks $= \frac{\text{period}}{4} = \frac{2\pi}{4} = \frac{\pi}{2}$

tick mark calculations:

(1) $\frac{-\pi}{4}$

(2) $\frac{-\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4}$

(3) $\frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$

(4) $\frac{3\pi}{4} + \frac{\pi}{2} = \frac{5\pi}{4}$

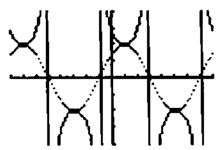
(5) $\frac{5\pi}{4} + \frac{\pi}{2} = \frac{7\pi}{4}$

endpoints Solve:

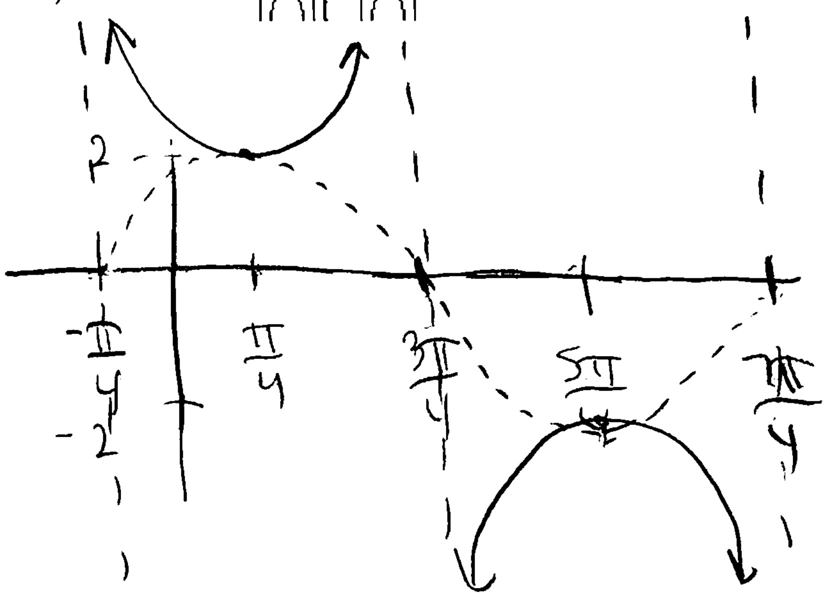
$bx - c = 0$	$bx - c = 2\pi$
$x + \frac{\pi}{4} = 0$	$x + \frac{\pi}{4} = 2\pi$
$x = \frac{-\pi}{4}$	$x \overset{4}{\underset{4}{=}} 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$
(starts)	(ends)

Remember to graph:

$y = 2\sin\left(x + \frac{\pi}{4}\right)$



vertical shift = d = none



Example: $y = \sec(2x)$ (Remember APTEV)

Formulas for General Form $y = a\sin(bx - c) + d$ **and** $y = a\cos(bx - c) + d$

amplitude $= |a| = |1| = 1$

period (of sine and cosine) =

$$\frac{2\pi}{b} = \frac{2\pi}{2} = \pi$$

tick marks $= \dfrac{\text{period}}{4} = \dfrac{\pi}{4}$

tick mark calculations:

(1) 0

(2) $0 + \frac{\pi}{4} = \frac{\pi}{4}$

(3) $\frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$

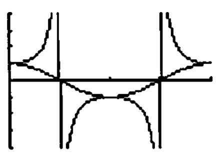
(4) $\frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$

(5) $\frac{3\pi}{4} + \frac{\pi}{4} = \pi$

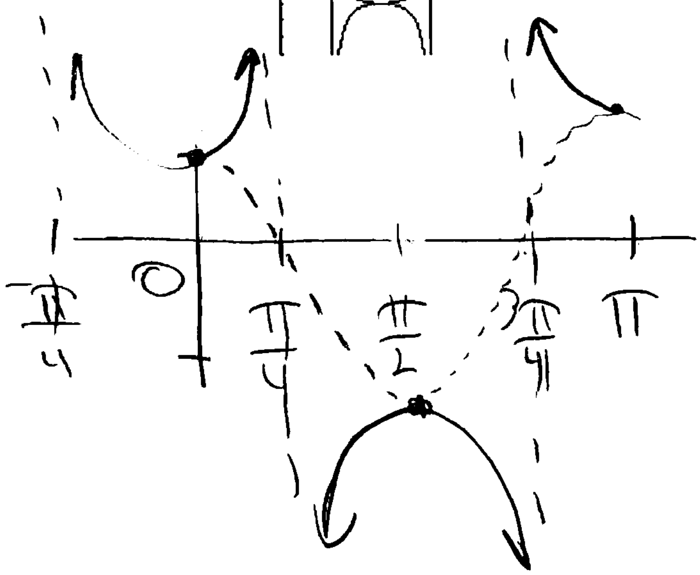
endpoints Solve:

$bx - c = 0$	$bx - c = 2\pi$
$2x = 0$	$2x = 2\pi$
$x = 0$	$x = \pi$
(starts)	(ends)

Remember to first graph:
$y = \cos(2x)$

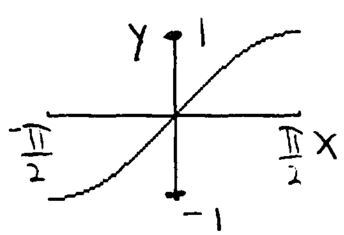
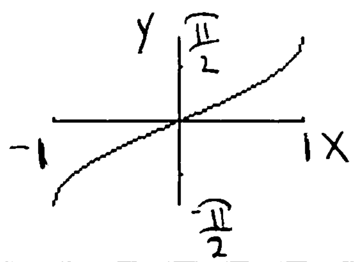
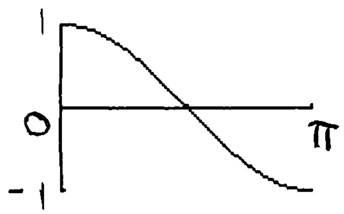
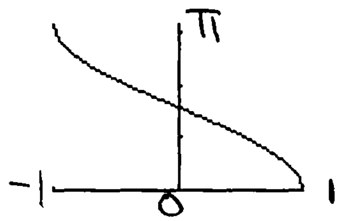
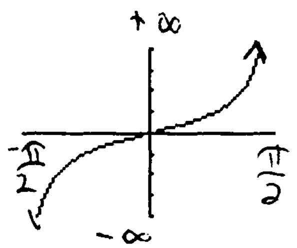
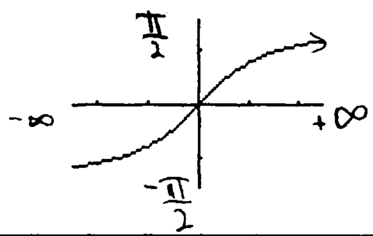


vertical shift = d = none



§7.1 Inverse Trigonometric Functions

- the function $y = \sin x$ is not one-to-one since its graph fails the horizontal line test
- by restricting the domain of $y = \sin x$ to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ it is one-to-one and has an inverse function $y = \arcsin x$

Function	Inverse Function	Restrictions
		$y = \arcsin x$ $y = \sin^{-1}(x)$ $x = \sin y$ $-1 \le x \le 1$ (domain) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ (range)
		$y = \arccos x$ $x = \cos y$ $-1 \le x \le 1$ (domain) $0 \le y \le \pi$ (range)
		$y = \arctan x$ $x = \tan y$ $-\infty < x < \infty$ (domain) $-\frac{\pi}{2} < y < \frac{\pi}{2}$ (range)

<u>Note</u>: Another notation for $y = \arcsin x$ is

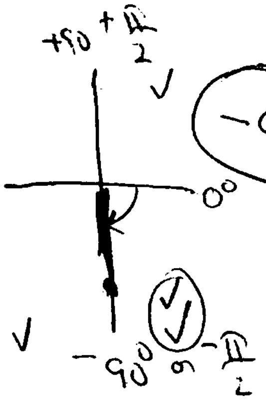
$$\sin^{-1} x \neq \frac{1}{\sin x} = (\sin x)^{-1}$$

<u>Note</u>: in $y = \sin^{-1} x$, y is the angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is x

Example 1 Evaluate.

$\boxed{\sin(x) = -1}$

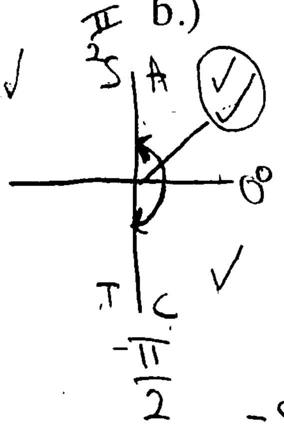
a.) $\arcsin(-1)$



-90° or $-\frac{\pi}{2}$

$\boxed{\sin(x) = +\frac{1}{2}}$

b.) $\sin^{-1}\left(\frac{1}{2}\right) = x$



30°

$-90° \le x \le +90°$

Domain

c.) $\sin^{-1}(2)$

$-1 \le x \le 1$

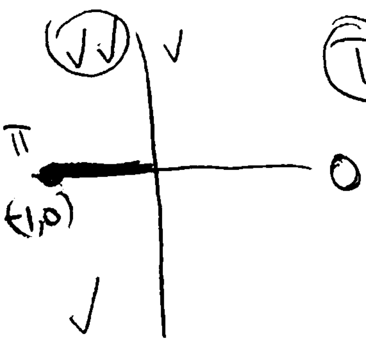
none

undefined

Example 2 Evaluate. $\cos(x) = \frac{-\sqrt{2}}{2}$

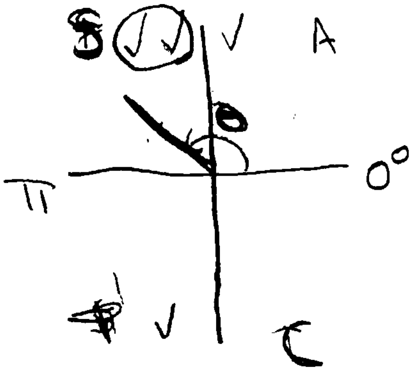
$\cos(x) = -1$

a.) $\cos^{-1}(-1)$



π

b.) $\arccos\left(\frac{-\sqrt{2}}{2}\right)$



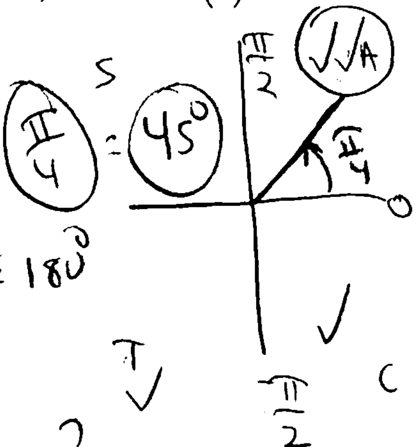
135°

$\frac{3\pi}{4}$

$0° \le y \le 180°$

$\tan(x) = +1$

c.) $\arctan(1)$



$\frac{\pi}{4} = 45°$

$\tan(x) = 1$?

$\frac{\sin}{\cos}$

Example 3 Evaluate. (use calculator)

a.) $\tan^{-1}(-8.45)$

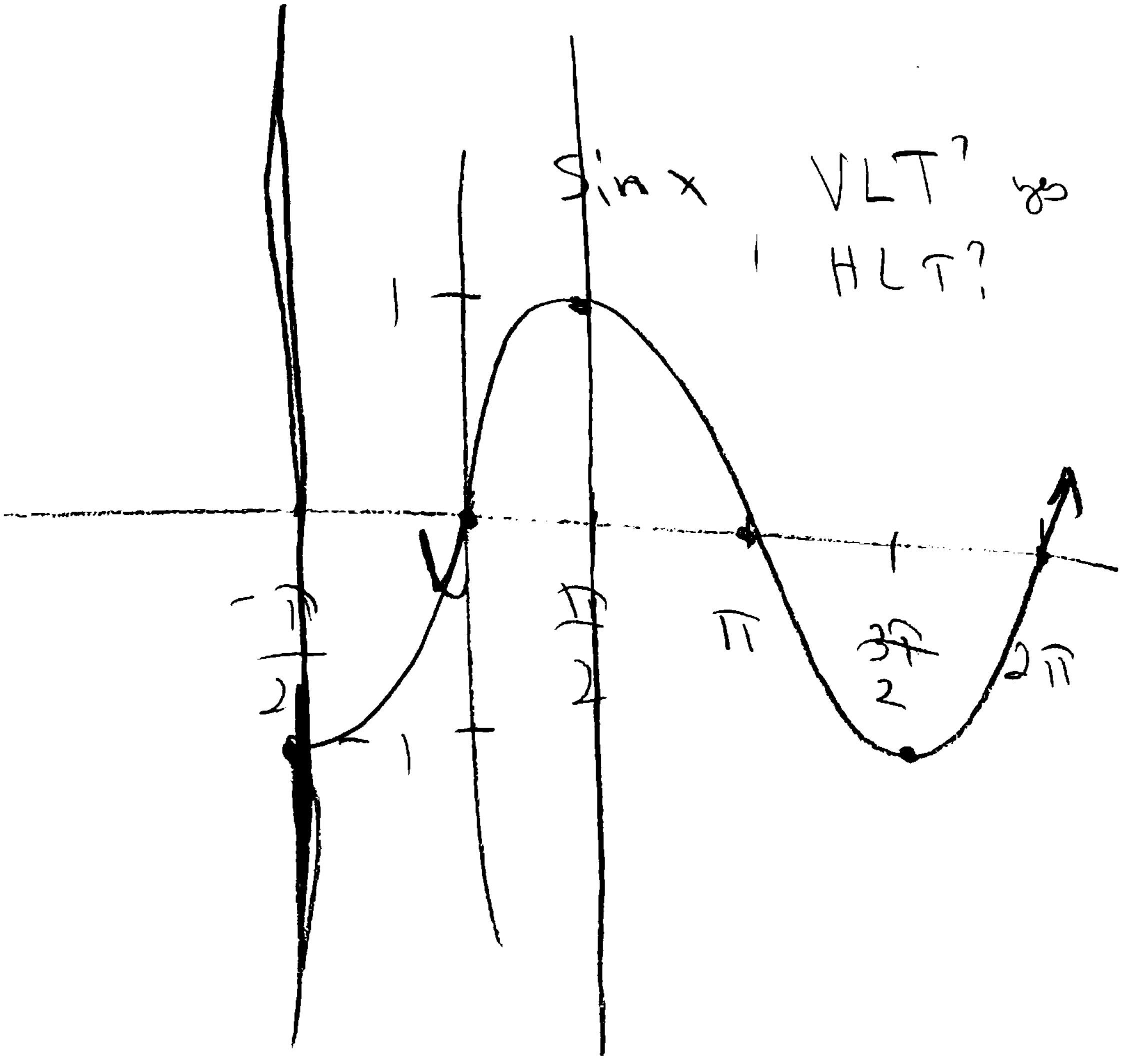
-83.25°

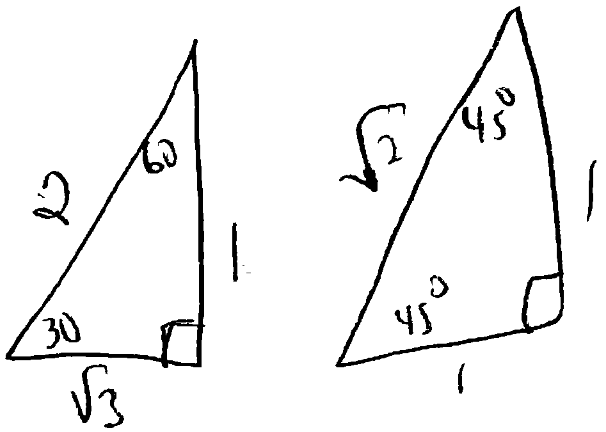
b.) $\sin^{-1}(0.2447)$

14.16°

c.) $\arccos(\cancel{2})$

undefined


Sin x
VLT? yes
HLT?
1
-1
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π


2
60
30
1
$\sqrt{3}$
$\sqrt{2}$
45^0
45^0
1
1

Inverse Functions?

$f(x) = 5x$

$g(x) = \frac{x}{5}$

$f(7) = 5(7) = 35$

$(7, 35)$

$g(35) = \frac{35}{5} = 7$

$(35, 7)$

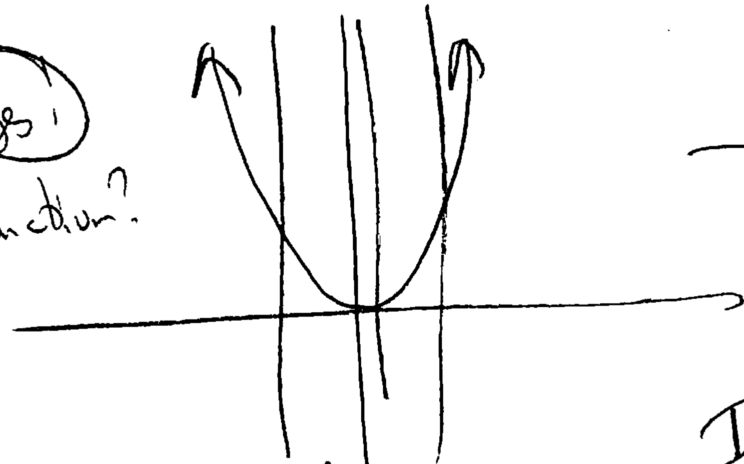
$\sin(30°) = \frac{1}{2}$

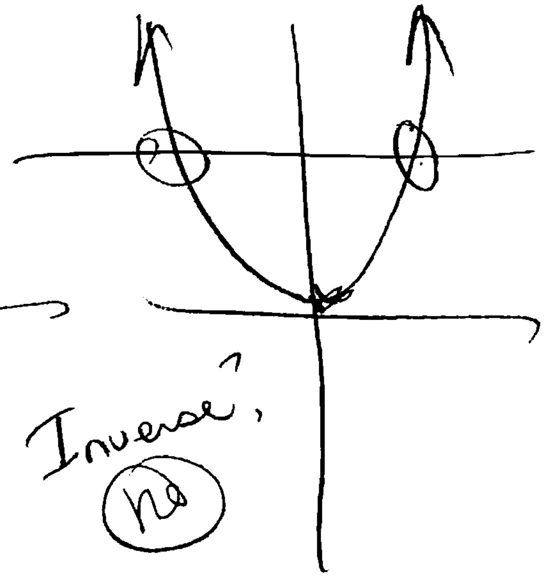
$\boxed{\sin(\measuredangle) = \#}$

$\sin^{-1}(\#) = \measuredangle$

yes!

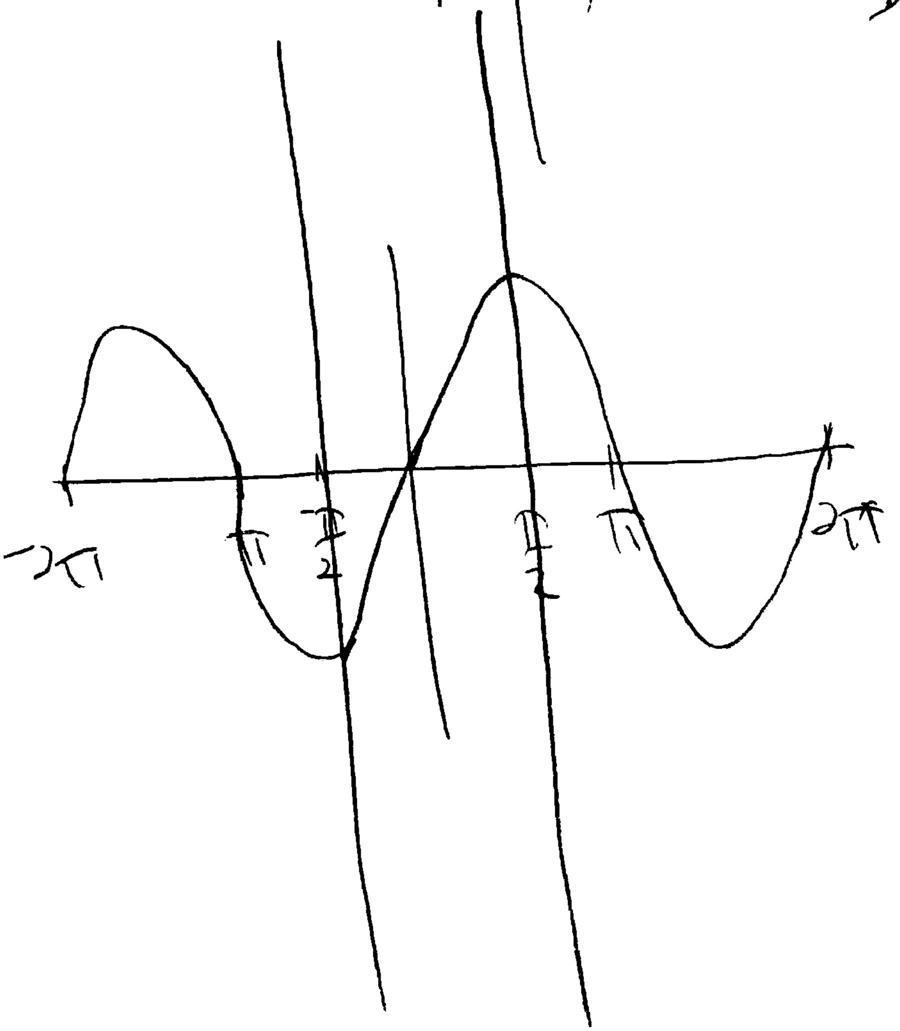
function?

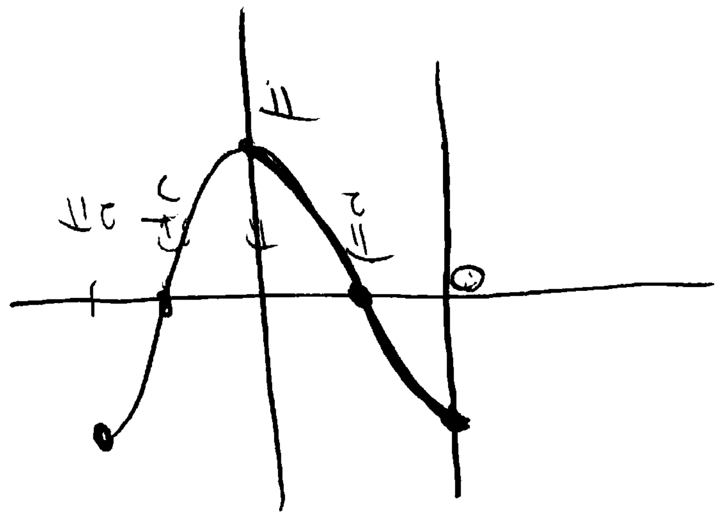




Inverse?

no





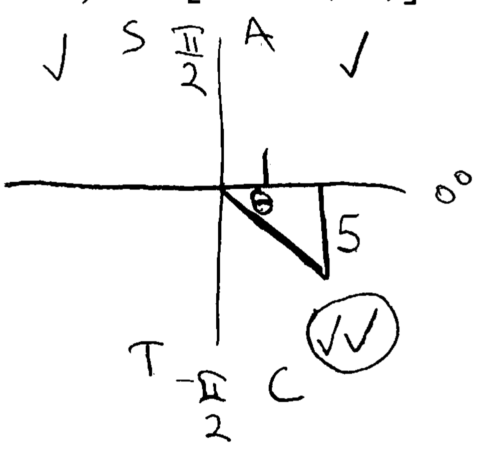
Compositions of Functions

If possible find the exact value.

Example 4 Find the radian value to three decimal places.

$-1 \le x \le 1$

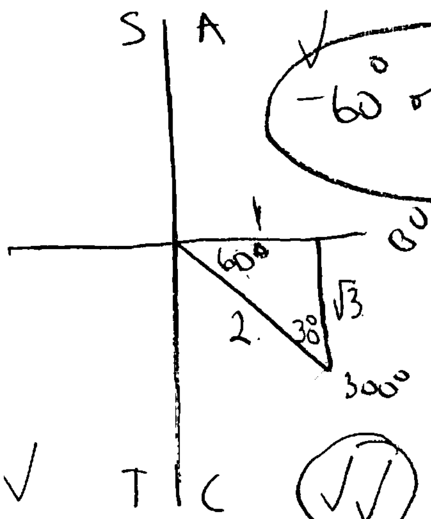
a.) $\tan[\arctan(-5)]$



-5

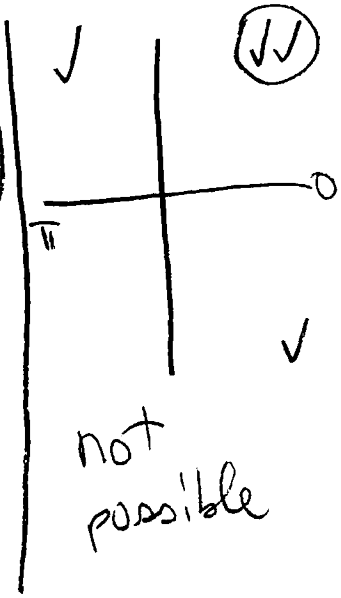
b.) $\arcsin\left(\sin\frac{5\pi}{3}\right)$

-60° or $-\frac{\pi}{3}$



$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \angle\,?$

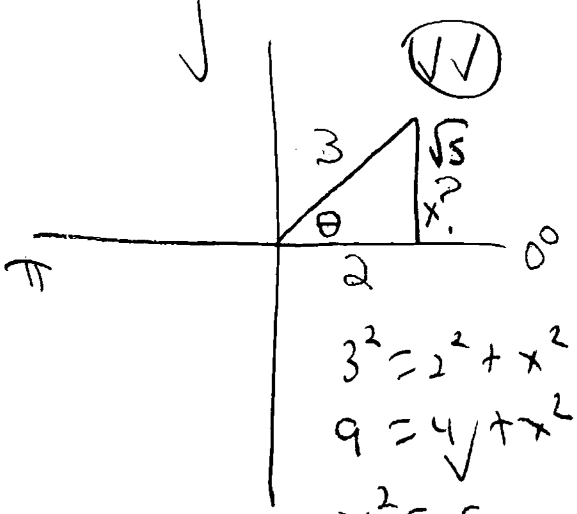
c.) $\cos\left(\cos^{-1}(\pi)\right)$



not possible

Example 5 Evaluate.

a.) $\tan\left(\arccos\frac{+2}{3}\right) = \frac{\sqrt{5}}{2}$



$3^2 = 2^2 + x^2$

$9 = 4 + x^2$

$x^2 = 5$

$x = \sqrt{5}$

b.) $\cos\left[\arcsin\left(\frac{-3}{5}\right)\right]$

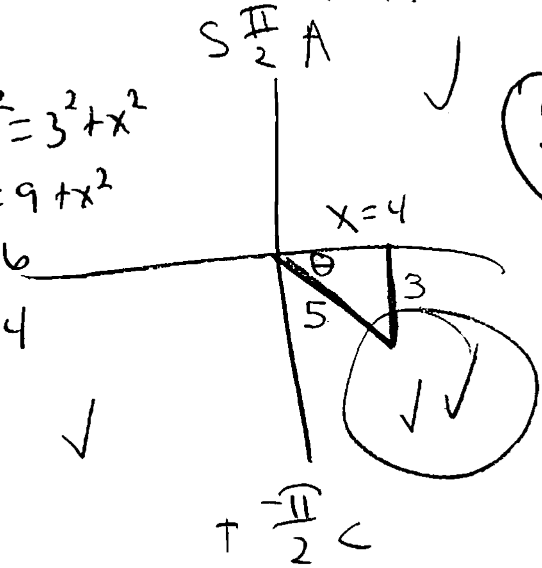
$\frac{4}{5}$

$5^2 = 3^2 + x^2$

$25 = 9 + x^2$

$x = 16$

$x = 4$



$$(f \circ g)(x) = f[g(x)]$$

$$= x$$

$$(g \circ f)(x) = x$$

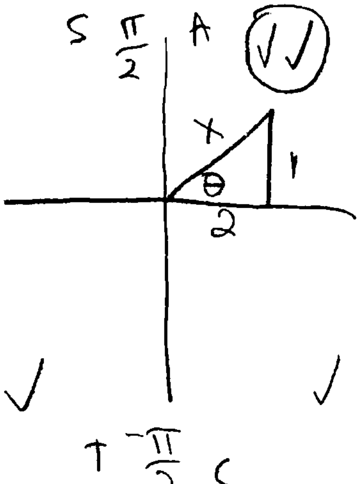
§7.2 Inverse Trigonometric Functions (cont.)

Compositions of Functions

If possible find the exact value.

Example

a.) $\sin\left[\tan^{-1}\left(\frac{1}{2}\right)\right]$



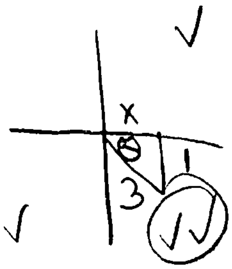
$x^2 = 2^2 + 1^2$

$x^2 = 5$

$x = \sqrt{5}$

$\sin\theta = \frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}$ $+\frac{\sqrt{5}}{5}$

b.) $\cos\left[\sin^{-1}\left(\frac{-1}{3}\right)\right]$



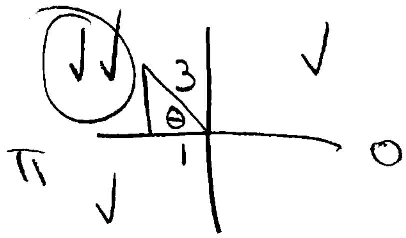
$3^2 = 1^2 + x^2$

$x^2 = 8$

$x = 2\sqrt{2}$

$\cos\theta = \frac{2\sqrt{2}}{3}$

c.) $\tan\left[\cos^{-1}\left(\frac{-1}{3}\right)\right]$



$3^2 = 1^2 + x^2$

$x = 2\sqrt{2}$

$\tan\theta = \frac{-2\sqrt{2}}{1}$

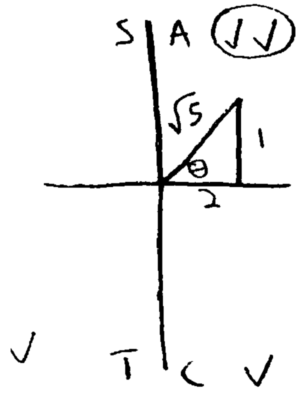
§7.2 Inverse Trigonometric Functions (cont.)

Compositions of Functions

If possible find the exact value.

Example

a.) $\sin\left[\tan^{-1}\frac{1}{2}\right]$



$1^2+2^2=x^2$

$5=x^2$

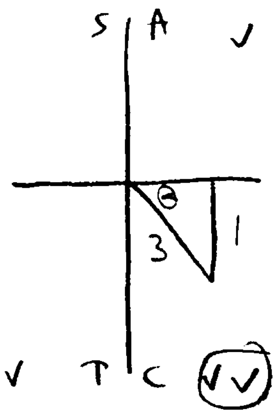
$x=\sqrt{5}$

$\tan^{-1}\left(\frac{1}{2}\right)=x$

$\sin(\theta)=\frac{1}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}}$

$=\frac{\sqrt{5}}{5}$

b.) $\cos\left[\sin^{-1}\left(\frac{-1}{3}\right)\right]$



$3^2=1^2+x^2$

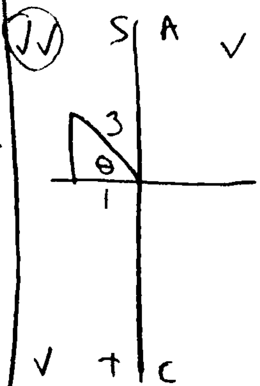
$x^2=8$

$x=2\sqrt{2}$

$\sin^{-1}\left(\frac{-1}{3}\right)=x$

$\cos\theta=\frac{2\sqrt{2}}{3}$

c.) $\tan\left[\cos^{-1}\left(\frac{-1}{3}\right)\right]$



$3^2=1^2+x^2$

$x^2=9$

$x=2\sqrt{2}$

$\cos^{-1}\left(\frac{-1}{3}\right)=x$

$\tan\theta=-\frac{2\sqrt{2}}{1}$

$x^2 - 1$

$(x+1)(x-1)$

$(\sin\theta)^2 = (\sin\theta)(\sin\theta)$

$= \sin^2\theta$

$\sin^2\theta + \cos^2\theta = 1$

$-\sin^2\theta \qquad -\sin^2 t$

$\cos^2\theta = \boxed{1 - \sin^2\theta}$

$$\frac{x}{x-2} + \frac{\lambda}{\lambda+1}$$

§7.3 Trigonometric Identities

Fundamental Trigonometric Identities

Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Quotient or Ratio Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

Cofunction Identities

$$\sin(90° - \theta) = \cos\theta \qquad \cos(90° - \theta) = \sin\theta \qquad \tan(90° - \theta) = \cot\theta$$

$$\csc(90° - \theta) = \sec\theta \qquad \sec(90° - \theta) = \csc\theta \qquad \cot(90° - \theta) = \tan\theta$$

Even and Odd Trigonometric Functions

The cosine and secant functions are **even**.

$\cos(-t) = \cos t$ $\qquad$ $\sec(-t) = \sec t$

The sine, cosecant, tangent, and cotangent functions are **odd**.

$\sin(-t) = -\sin(t)$ $\qquad$ $\csc(-t) = -\csc(t)$

$\tan(-t) = -\tan(t)$ $\qquad$ $\cot(-t) = -\cot(t)$

$$\cancel{2x}+5 = \cancel{2x}+5$$
$$-\cancel{2x} \qquad -\cancel{2x}$$

$$5 = 5$$

identity

$$\mathbb{R}$$

$$(1+x)(1-x)$$

$$1^2 - x^2$$

$$1 - x^2$$

Example 1 Simplify : $\frac{\cot\theta}{\csc\theta} = \frac{\frac{\cos\theta}{\sin\theta}}{\frac{1}{\sin\theta}}$

$\frac{\cos\theta}{\cancel{\sin\theta}} \cdot \frac{\cancel{\sin\theta}}{1} = \cos\theta$

Example 2 Simplify: $\frac{\cos\theta}{(1+\sin\theta)} \cdot \frac{(1-\sin\theta)}{(1-\sin\theta)}$

$= \frac{\cos\theta - \cos\theta\sin\theta}{1 - \cancel{\sin\theta} + \cancel{\sin\theta} - \sin^2\theta} = \frac{\cos\theta(1-\sin\theta)}{1-\sin^2\theta}$

$= \frac{\cos\theta(1-\sin\theta)}{\cos^2\theta} = \frac{\cancel{\cos\theta}(1-\sin\theta)}{(\cancel{\cos\theta})(\cos\theta)} = \boxed{\frac{1-\sin\theta}{\cos\theta}}$

Example 3 Simplify: $\frac{1+\sin u}{\sin u} + \frac{\cot u - \cos u}{\cos u}$

LCD = $(\sin u)(\cos u)$

$\frac{1+\sin u}{\sin u} \cdot \frac{\cos u}{\cos u} = \frac{\cos u(1+\sin u)}{\cos u \sin u}$

$\frac{\cot u - \cos u}{\cos u} \cdot \frac{\sin u}{\sin u} = \frac{\sin u(\cot u - \cos u)}{\cos u \sin u}$

$\frac{\cos u + \cancel{\cos u \sin u} + \sin u \cdot \cot u - \cancel{\sin u \cos u}}{\cos u \sin u} = \frac{\cos u + \cos u}{\cos u \sin u}$

$= \frac{2\cancel{\cos u}}{\cancel{\cos u}\sin u} = \boxed{\frac{2}{\sin u}} = \boxed{2\csc u}$

$$\frac{\frac{x}{y}}{\frac{1}{y}} = \frac{x}{y} \cdot \frac{y}{1} = x$$

$$\frac{\frac{x}{y} \cdot y}{\frac{1}{y} \cdot \frac{y}{1}} = \frac{x}{1} = x$$

Example 4 Simplify: $\dfrac{\sin^2\theta - 1}{\tan\theta\sin\theta - \tan\theta}$

$$= \frac{\cancel{(\sin\theta - 1)}(\sin\theta + 1)}{\tan\theta\cancel{(\sin\theta - 1)}} = \boxed{\frac{\sin\theta + 1}{\tan\theta}}$$

Establish (prove)Trigonometric Identities

- an <u>identity</u> is an equation which is true for all values for which the equation is defined
- to verify an identity, generally work with one side of the equation and show that it equals the other side
- some suggestions to consider when verifying identities:

1.) simplify the more complex side

2.) perform algebraic operations including squaring, factoring, adding or subtracting fractions, multiplying the numerator and denominator by a nonzero factor

3.) rewrite in terms of sine and cosine

4.) rewrite in terms of a single trigonometric function

5.) use other identities (reciprocal identities, ratio identities, Pythagorean identities)

<u>Examples</u> Verify each identity.

a.) $\csc\theta \cdot \tan\theta = \sec\theta$

$$\frac{1}{\cancel{\sin\theta}} \cdot \frac{\cancel{\sin\theta}}{\cos\theta} = \frac{1}{\cos\theta} = \sec\theta$$

b.) $\sin^2(-\theta)+\cos^2(-\theta)=1$

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$(-\sin(\theta))^2 + (\cos\theta)^2 = 1$

$\sin^2\theta + \cos^2\theta = 1$

c.) $\dfrac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)} = \cos\theta - \sin\theta$

$$\frac{\sin^2\theta - \cos^2\theta}{-\sin\theta - \cos\theta} = \frac{(\sin\theta - \cos\theta)\cancel{(\sin\theta + \cos\theta)}}{-\cancel{(\sin\theta + \cos\theta)}}$$

$$= \frac{\sin\theta - \cos\theta}{-1}$$

$$= -\sin\theta + \cos\theta$$

$$= \boxed{\cos\theta - \sin\theta}$$

d.) $\dfrac{1+\tan u}{1+\cot u}=\tan u$

$$\frac{\left(\frac{1}{1}+\frac{\sin u}{\cos u}\right)}{\left(\frac{1}{1}+\frac{\cos u}{\sin u}\right)}\cdot\frac{(\cos u \sin u)}{(\cos u \sin u)}=\frac{\cos u \sin u+\sin^2 u}{\cos u \sin u+\cos^2 u}$$

$$=\frac{\sin u\,\cancel{(\cos u+\sin u)}}{\cos u\,\cancel{(\cos u+\sin u)}}=\frac{\sin u}{\cos u}=\tan u$$

LCD: $\sin\theta(1+\cos\theta)$

e.) $\dfrac{\sin\theta}{(1+\cos\theta)}+\dfrac{1+\cos\theta}{\sin\theta}=2\csc\theta$

$$\frac{\sin\theta}{1+\cos\theta}\cdot\frac{\sin\theta}{\sin\theta}=\frac{\sin^2\theta}{\sin\theta(1+\cos\theta)}$$

$$\frac{1+\cos\theta}{\sin\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta}=\frac{1+2\cos\theta+\cos^2\theta}{\sin\theta(1+\cos\theta}$$

$$\Rightarrow\frac{(\sin^2\theta)+(1+2\cos\theta+\cos^2\theta)}{\sin\theta(1+\cos\theta)}=\frac{1+1+2\cos\theta}{\sin(1+\cos\theta)}$$

$$=\frac{2+2\cos\theta}{\sin(1+\cos\theta)}=\frac{2\cancel{(1+\cos\theta)}}{\sin\cancel{(1+\cos\theta)}}=\frac{2}{\sin\theta}=2\frac{1}{\sin\theta}$$

$$=\boxed{2\csc\theta}$$

f.) $\dfrac{\tan v + \cot v}{\sec v \csc v} = 1$

$$\frac{\left(\frac{\sin v}{\cos v} + \frac{\cos v}{\sin v}\right)}{\left(\frac{1}{\cos v} \cdot \frac{1}{\sin v}\right)} \frac{(\cos v \sin v)}{\left(\frac{\cancel{\cos v}\ \cancel{\sin v}}{1}\right)} = \frac{\sin^2 v + \cos^2 v}{1}$$

$$= \frac{1}{1} = 1$$

§7.4 Sum and Difference Formulas

REMEMBER YOU KNOW ALGEBRA !

Sum or Difference of Two Angles Identities
$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$
$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$
$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$
$\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
$\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

Cofunction Identities	
$\sin(90^\circ-\theta)=\cos\theta$	$\cos(90^\circ-\theta)=\sin\theta$
$\tan(90^\circ-\theta)=\cot\theta$	$\cot(90^\circ-\theta)=\tan\theta$
$\sec(90^\circ-\theta)=\csc\theta$	$\csc(90^\circ-\theta)=\sec\theta$

Example 1 Find the exact value. $\cos 75^\circ$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(30^\circ+45^\circ) = \cos 30^\circ \cos 45^\circ - \sin 30^\circ \sin 45^\circ$$

$$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \boxed{\frac{\sqrt{6}-\sqrt{2}}{4}}$$

Example 2 Find the exact value. $\cos\frac{\pi}{12}$

$\frac{\pi}{12} \cdot \frac{180}{\pi} = 15^\circ$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\cos(45^\circ-30^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}$$

Example 3 Find the exact value.

$\sin 80^\circ \cos 20^\circ - \cos 80^\circ \sin 20^\circ$

$$\sin(80^\circ - 20^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

Example 4 Prove a Cofunction $\cos\left(\frac{\pi}{2} - x\right)$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\cos\left(\frac{\pi}{2} - x\right) = \cos 90^\circ \cos x + \sin 90^\circ \sin x$$

$$= (0)\cos x + 1(\sin x)$$

$$= 0 + \sin x = \boxed{\sin x}$$

Example 5 Simplify. (Identity)

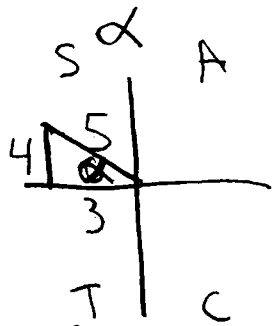
$$\boxed{\tan(\theta + \pi) = \tan\theta}$$

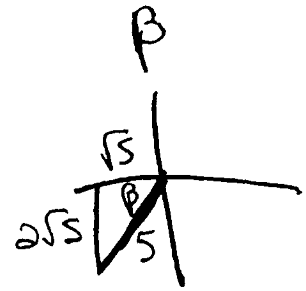
$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{\tan\theta + \tan\pi}{1 - \tan\theta\tan\pi}$$

$$= \frac{\tan\theta + 0}{1 - \tan\theta(0)} = \frac{\tan\theta}{1}$$

Example 6 Given $\sin\alpha = \frac{4}{5}$, α in quadrant II, and $\sin\beta = \frac{-2\sqrt{5}}{5}$, in quadrant III, find $\cos(\alpha + \beta)$





$$5^2 = 4^2 + x^2$$
$$25 = 16 + x^2$$
$$x^2 = 9 \quad x = 3$$

$$5^2 = (2\sqrt{5})^2 + x^2$$
$$25 = 20 + x^2$$
$$x^2 = 5 \quad x = \sqrt{5}$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$= \left(-\frac{3}{5}\right)\left(-\frac{\sqrt{5}}{5}\right) - \left(\frac{4}{5}\right)\left(\frac{-2\sqrt{5}}{5}\right)$$

$$= \frac{3\sqrt{5}}{25} + \frac{8\sqrt{5}}{25} = \boxed{\frac{11\sqrt{5}}{25}}$$

$$\sin u \cot u = \frac{\cancel{\sin u}}{1} \cdot \frac{\cos u}{\cancel{\sin u}}$$

$$= \cos u$$

$$\frac{2}{x} = \frac{2}{1} \cdot \frac{1}{x}$$

$$= 2 \cdot \frac{1}{\sin u} \qquad \frac{2\cancel{x}}{\cancel{x}} = 2$$

$$= \boxed{2 \csc u}$$

§7.5 Double-Angle and Half –Angle Formulas

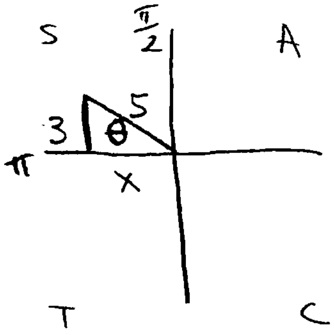
REMEMBER YOU KNOW ALGEBRA !

Double-Angle Identities	
$\sin 2\alpha = 2\sin\alpha\cos\alpha$	$\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$
$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$ $= 1 - 2\sin^2\alpha$ $= 2\cos^2\alpha - 1$	

Half-Angle Identities	
$\sin\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\alpha}{2}}$	$\tan\dfrac{\alpha}{2} = \dfrac{\sin\alpha}{1+\cos\alpha} = \dfrac{1-\cos\alpha}{\sin\alpha}$
$\cos\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1+\cos\alpha}{2}}$	

Example 1 Evaluate. Find $\sin 2\theta$, $\cos 2\theta$, $\tan 2\theta$

from $\sin\theta = \dfrac{3}{5}$, $\dfrac{\pi}{2} < \theta < \pi$



$5^2 = 3^2 + x^2$

$25 = 9 + x^2$

$16 = x^2$

$x = 4$

$\sin 2\theta = 2\sin\theta\cos\theta$

$= \left(\frac{2}{1}\right)\left(\frac{3}{5}\right)\left(\frac{-4}{5}\right)$

$= \frac{-24}{25}$

$\cos\theta = \frac{adj}{hyp} = \frac{-4}{5}$ $\tan\theta = \frac{-3}{4}$

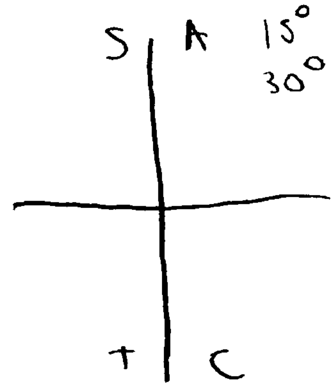
$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(-\frac{3}{4}\right)}{1-\left(-\frac{3}{4}\right)^2} = \frac{-\frac{6}{4}}{\frac{7}{16}} = \frac{-6}{\cancel{4}} \cdot \frac{\cancel{16}^{4}}{7}$ $\boxed{\frac{-24}{7}}$

$\cos 2\theta = 1 - 2\sin^2\theta$

$= 1 - 2\left(\frac{3}{5}\right)^2 = 1 - \frac{2}{1}\left(\frac{9}{25}\right) = 1 - \frac{18}{25} = \frac{25}{25} - \frac{18}{25} = \boxed{\frac{7}{25}}$

Example 2 Find the exact value. $\cos 15^\circ$

$\alpha = 30^\circ$ $\frac{\alpha}{2} = \frac{30^\circ}{2} = 15^\circ$



$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$

$\cos\frac{30^\circ}{2} = \pm\sqrt{\frac{1+\cos 30^\circ}{2}}$

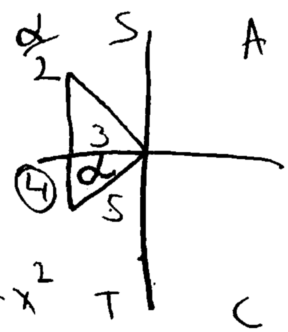
$= +\sqrt{\frac{\frac{1}{1}+\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{\frac{\sqrt{3}+2}{2}}{\frac{2}{1}}}$

$= \sqrt{\frac{\sqrt{3}+2}{2}\cdot\frac{1}{2}} = \sqrt{\frac{\sqrt{3}+2}{4}} = \frac{\sqrt{\sqrt{3}+2}}{\sqrt{4}} = \boxed{\frac{\sqrt{\sqrt{3}+2}}{2}}$

$90^\circ < \frac{\alpha}{2} < 135^\circ$

Example 3 If $\cos\alpha = \frac{-3}{5}$, $\pi < \alpha < \frac{3\pi}{2}$,

find the exact value of;



$5^2 = 3^2 + x^2$

a) $\sin\frac{\alpha}{2}$

$= \pm\sqrt{\frac{1-\cos\alpha}{2}}$

$= +\sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}}$

$= +\sqrt{\frac{\frac{5}{5}+\frac{3}{5}}{2}}$

$+\sqrt{\frac{\frac{8}{5}}{\frac{2}{1}}}$ $\quad \frac{\cancel{8}^{4}}{5}\cdot\frac{1}{\cancel{2}}$

$+\sqrt{\frac{4}{5}}$ $= +\frac{\sqrt{4}}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}}$

$= +\frac{2\sqrt{5}}{5}$

b) $\cos\frac{\alpha}{2}$

$= \pm\sqrt{\frac{1+\cos\alpha}{2}}$

$= -\sqrt{\frac{1-\frac{3}{5}}{2}}$

$= -\sqrt{\frac{\frac{5}{5}-\frac{3}{5}}{2}}$

$= -\sqrt{\frac{\frac{2}{5}}{\frac{2}{1}}}$

$= -\sqrt{\frac{1}{5}}$ $= -\frac{\sqrt{1}}{\sqrt{5}}$

$= -\frac{\sqrt{5}}{5}$

c) $\tan\frac{\alpha}{2}$

$= \frac{1-\cos\alpha}{\sin\alpha}$

$= \frac{1-\left(-\frac{3}{5}\right)}{-\frac{4}{5}}$

$= \frac{\frac{5}{5}+\frac{3}{5}}{-\frac{4}{5}}$

$= \frac{\frac{8}{5}}{-\frac{4}{5}}$

$= \frac{8}{5}\cdot -\frac{5}{4}$

$= -2$

§7.6 Product-to-Sum & Sum-to-Product Formulas

REMEMBER YOU KNOW ALGEBRA !

Product to Sum Identities
$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$
$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$
$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$
$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$

Example 1 Rewrite as a sum or difference.

a) $\sin(6\theta)\sin(4\theta)$

$\sin\alpha \sin\beta$

$= \frac{1}{2}[\cos(6\theta-4\theta) - \cos(6\theta+4\theta)]$

$= \boxed{\frac{1}{2}[\cos(2\theta) - \cos(10\theta)]}$

b) $\sin(\overset{\alpha}{3\theta})\cos(\overset{\beta}{5\theta})$

$= \frac{1}{2}[\sin(3\theta+5\theta)+\sin(3\theta-5\theta)]$

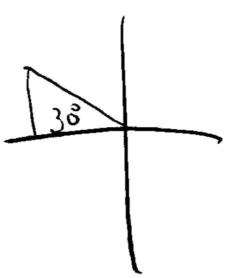
$= \frac{1}{2}[\sin(8\theta) + \sin(-2\theta)]$

$= \frac{1}{2}[\sin(8\theta) - \sin(2\theta)]$

Sum-to-Product Identities
$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$
$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$
$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$
$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

Example 2 Find the exact value of $\cos \overset{x}{195^\circ} + \cos \overset{y}{105^\circ}$

$$= 2\cos\left(\frac{195^\circ+105^\circ}{2}\right)\cos\left(\frac{195^\circ-105^\circ}{2}\right)$$



$$= 2\cos\left(\frac{300^\circ}{2}\right)\cos\left(\frac{90^\circ}{2}\right)$$

$$= 2\cos(150^\circ)\cos(45^\circ) = \frac{\cancel{2}}{1}\left(\frac{-\sqrt{3}}{\cancel{2}}\right)\left(\frac{\sqrt{2}}{2}\right) = \boxed{\frac{-\sqrt{6}}{2}}$$

Example 3 Express as a product: $\cos(\overset{x}{3\theta}) + \cos(\overset{y}{2\theta})$

$$= 2\cos\left(\frac{3\theta+2\theta}{2}\right)\cos\left(\frac{3\theta-2\theta}{2}\right) = \boxed{2\cos\left(\frac{5\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}$$

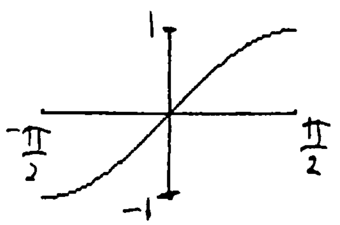
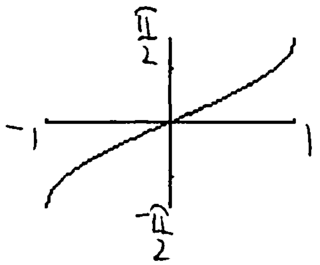
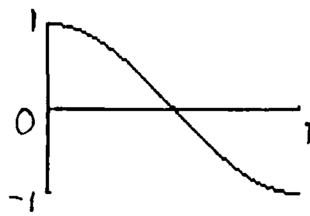
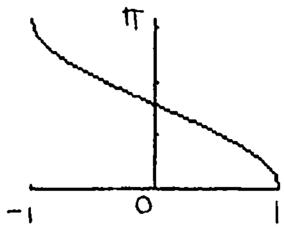
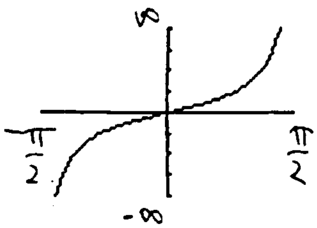
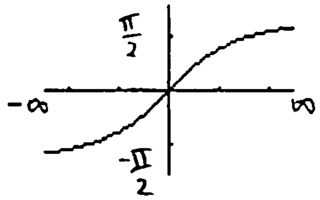
$$\sin 30^\circ = \frac{1}{2}$$

$$\sin \theta = \frac{1}{2}$$

§4.7 Inverse Trigonometric Functions

- the function $y = \sin x$ is not one-to-one since its graph fails the horizontal line test

- by restricting the domain of $y = \sin x$ to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, it is one-to-one and has an inverse function $y = \arcsin x$

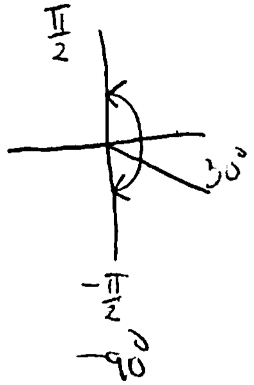
Function	Inverse Function	Restrictions
		$y = \arcsin x$ $x = \sin y$ $-1 \le x \le 1$ (domain) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ (range) $-90^\circ \le y \le 90^\circ$
		$y = \arccos x$ $x = \cos y$ $-1 \le x \le 1$ (domain) $0 \le y \le \pi$ (range)
		$y = \arctan x$ $x = \tan y$ $-\infty < x < \infty$ (domain) $-\frac{\pi}{2} < y < \frac{\pi}{2}$ (range)

<u>Note</u>: Another notation for $y = \arcsin x$ is $\sin^{-1} x \neq \dfrac{1}{\sin x} = (\sin x)^{-1}$

<u>Note</u>: in $y = \sin^{-1} x$, y is the <u>angle</u> in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is x.

Example 1 Evaluate.

a.) $\arcsin\left(\frac{-1}{2}\right)$



$\sin 30° = \frac{1}{2}$

$y = \arcsin\left(-\frac{1}{2}\right) = -30°$

b.) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\boxed{y = 60°}$

$\sin 60° = \frac{\sqrt{3}}{2}$

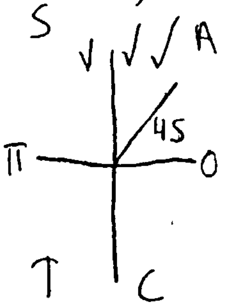
c.) $\sin^{-1}(2)$

undefined

$-1 \le x \le 1$
must be true!

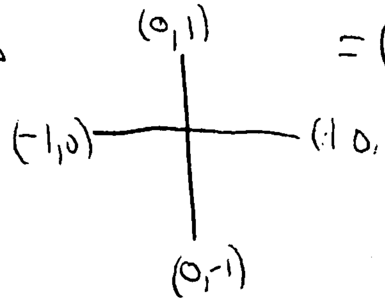
Example 2 Evaluate.

a.) $\arccos\left(\frac{\sqrt{2}}{2}\right)$



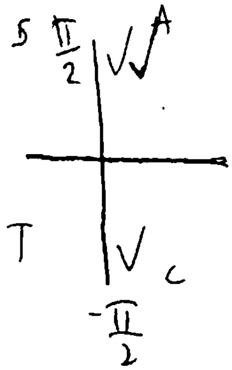
$= \boxed{45°}$

b.) $\cos^{-1}(-1)$



$= \pi$

c.) $\arctan(0)$



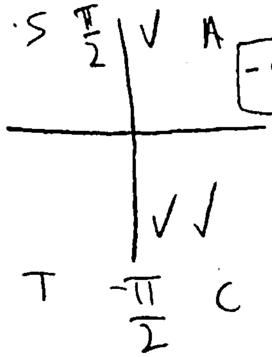
$\sin\theta = 0$

0°

$\tan\theta = 0$

Example 3 Evaluate. (use calculator)

a.) $\tan^{-1}(-8.45)$



$\boxed{-83.251°}$

b.) $\sin^{-1}(0.2447)$

$\boxed{14.160°}$

c.) $\arccos(2)$

not possible

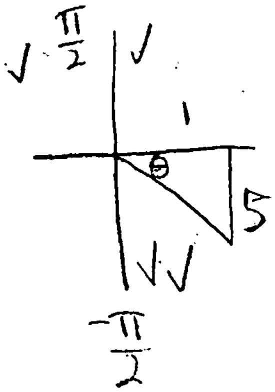
Compositions of Functions

$\frac{5\pi}{3} \cdot \frac{180}{\pi} = 300°$

If possible find the exact value.

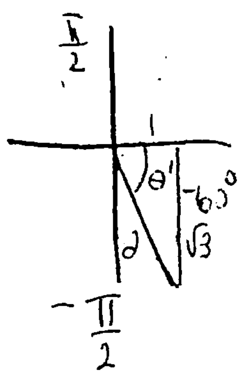
Example 4 Find the radian value to four decimal places.

a.) $\tan[\arctan(-5)]$



-5

b.) $\arcsin\left(\sin\frac{5\pi}{3}\right)$



$5(60°) = 300°$

$\arcsin\left(\frac{\sqrt{3}}{2}\right)$

$= -\frac{\pi}{3} = -60°$

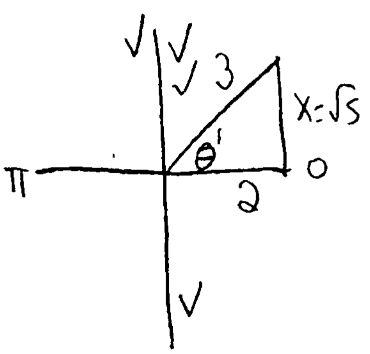
c.) $\cos(\cos^{-1}\pi)$

not possible

$-1 \le x \le 1$

Example 5 Evaluate.

a.) $\tan\left(\arccos\frac{2}{3}\right)$



$3^2 = 2^2 + x^2$

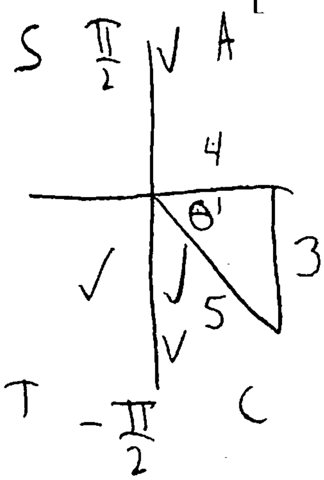
$9 = 4 + x^2$

$x^2 = 5$

$x = \sqrt{5}$

$= \frac{\sqrt{5}}{2}$

b.) $\cos\left[\arcsin\left(\frac{-3}{5}\right)\right]$



$5^2 = 3^2 + x^2$

$25 = 9 + x^2$

$x^2 = 16$

$x = 4$

$= \frac{4}{5}$

§7.7 Solving Trigonometric Equations (I)

REMEMBER YOU KNOW ALGEBRA !

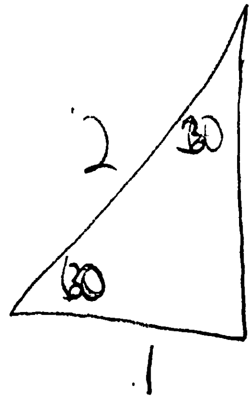
Example 1 Determine if $\theta = \frac{\pi}{4}$ is a solution of the equation $\sin\theta = \frac{1}{2}$. Is $\theta = \frac{\pi}{6}$ a solution ?

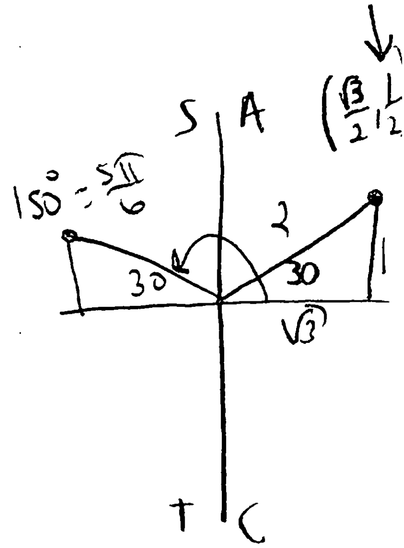
$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$!

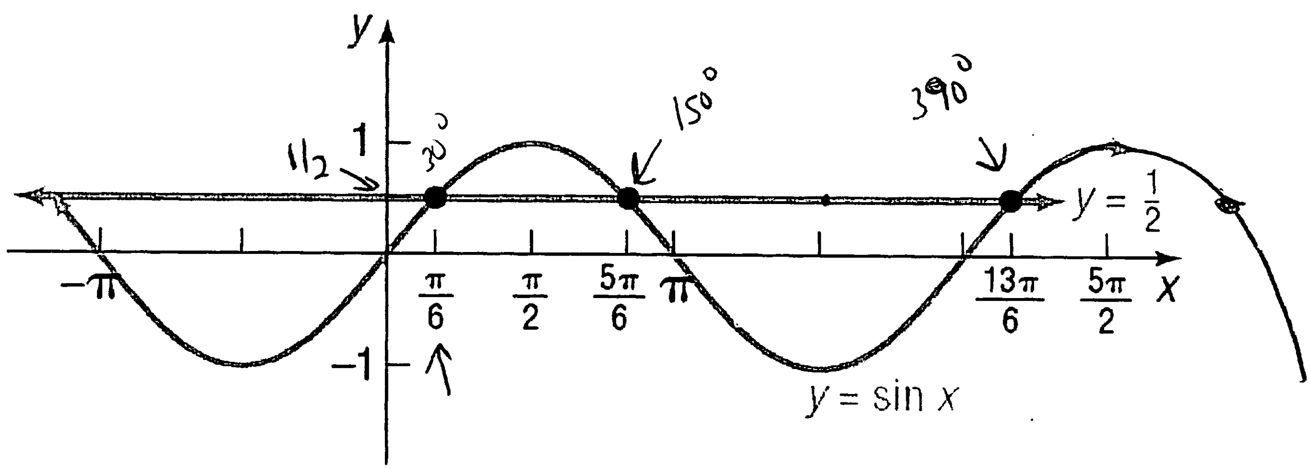
$\sin\frac{\pi}{6} = \frac{1}{2}$?

$\sin 30^\circ = \frac{1}{2}$?

$\sin\frac{5\pi}{6} = \frac{1}{2}$



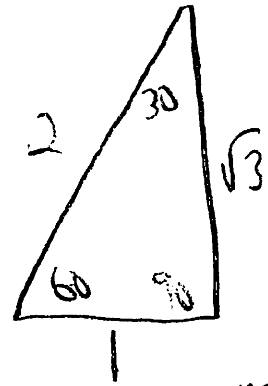


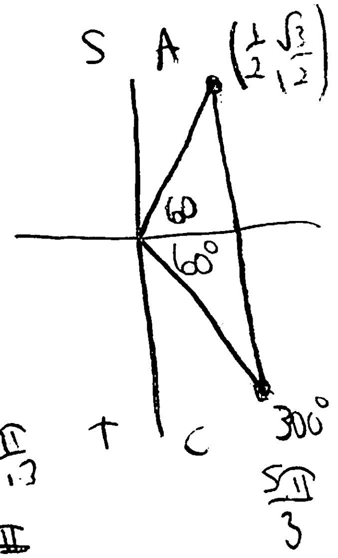


Example 2 Solve the equation: $\cos\theta = \dfrac{1}{2}$

Give a general formula for **ALL** the solutions.
List 8 of the solutions.

$\cos\theta = \frac{1}{2}$ $\theta = ?$





$\boxed{\theta = \frac{\pi}{3} + 2n\pi}$

$\boxed{\theta = \frac{5\pi}{3} + 2n\pi}$

$\frac{13\pi}{3} + \frac{\pi}{3}$ T

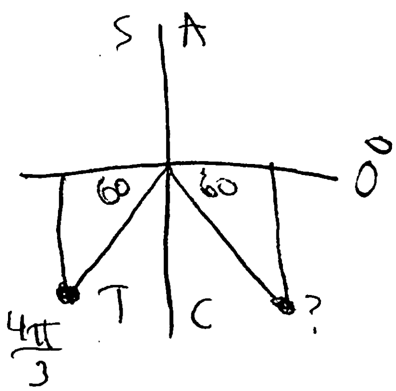
$\frac{\pi}{3}, \quad \frac{\pi}{3} + \frac{6\pi}{3} = \frac{7\pi}{3}, \quad \frac{13\pi}{3},$

Example 3 Solve the equation: $2\sin\theta + \sqrt{3} = 0$,

$\boxed{0 \le \theta < 2\pi}$

$\frac{2\sin\theta}{2} = \frac{-\sqrt{3}}{2}$

$\sin\theta = \frac{-\sqrt{3}}{2}$



$\theta = 240°, \boxed{\frac{4\pi}{3}}$

$\theta = 300°, \boxed{\frac{5\pi}{3}}$

$\theta = \frac{\pi}{3} + 2n\pi$

$n = 0$

$\frac{\pi}{3} + 0 = \boxed{\frac{\pi}{3}}$

$n = 1$

$\frac{\pi}{3} + \frac{6\pi}{3} = \boxed{\frac{7\pi}{3}}$

$n = 2$

$\frac{\pi}{3} + 2(2)\pi$

$\frac{\pi}{3} + \frac{4\pi}{1}$

$\frac{\pi}{3} + \frac{12\pi}{3} = \boxed{\frac{13\pi}{3}}$

$\theta = \frac{5\pi}{3} + 2n\pi$

$n = 0$

$\frac{5\pi}{3}$

$n = 1$

$\frac{5\pi}{3} + 2\pi$

$\frac{5\pi}{3} + \frac{6\pi}{3} = \frac{11\pi}{3}$

$n = 2$

$\frac{5\pi}{3} + 4\pi$

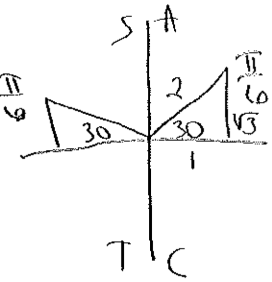
$\frac{5\pi}{3} + \frac{12\pi}{3} = \frac{17\pi}{3}$

Example 4 Solve the equation: $\sin(2\theta) = \frac{1}{2}$,

$\boxed{0 \le \theta < 2\pi}$

temp

$\sin\theta = \frac{1}{2}$



$2\theta = \frac{\pi}{6} + 2n\pi$ | $2\theta = \frac{5\pi}{6} + 2n\pi$

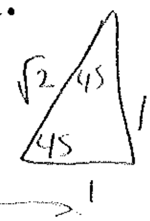
$\theta = \frac{\pi}{12} + n\pi$ | $\theta = \frac{5\pi}{12} + n\pi$

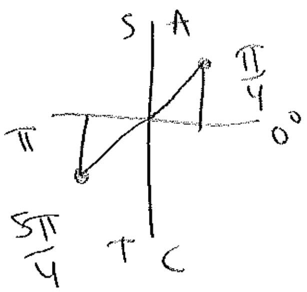
$n=0$: $\frac{\pi}{12}, \frac{5\pi}{12}$ | $n=1$: $\frac{\pi}{12} + \frac{12\pi}{12} = \frac{13\pi}{12}$, $\frac{5\pi}{12} + \frac{12\pi}{12} = \frac{17\pi}{12}$ | $\cancel{n=2}$: $\cancel{\frac{\pi}{12} + 2\pi}$

Example 5 Solve the equation: $\tan\left(\theta - \frac{\pi}{2}\right) = 1$,

$\boxed{0 \le \theta < 2\pi}$

$\tan\theta = 1$?





$\theta = \frac{\pi}{4}$ | $\theta = \frac{5\pi}{4}$ $\Rightarrow$ $\theta = \frac{\pi}{4} + n\pi$

$\theta - \frac{\pi}{2} = \frac{\pi}{4} + n\pi$

$\theta = \frac{\pi}{4} + \frac{\pi}{2} + n\pi$

$\theta = \frac{3\pi}{4} + n\pi$

$n=0$: $\theta = \frac{3\pi}{4}$

$n=1$: $\theta = \frac{3\pi}{4} + \frac{4\pi}{4} = \frac{7\pi}{4}$

$n=2$: $\theta = \frac{3\pi}{4} + 2\pi = \frac{3\pi}{4} + \frac{8\pi}{4} = \cancel{\frac{11\pi}{4}}$

$2x^2 - 3x + 1 = 0$?

$(2x - 1)(x - 1) = 0$

$2x - 1 = 0 \quad \Big| \quad x - 1 = 0$

$x = \frac{1}{2} \quad \Big| \quad x = 1$

§7.8 Solving Trigonometric Equations (II)

REMEMBER YOU KNOW ALGEBRA !

Example 1 Solve the equation: (Quadratic in Form)

$2\sin^2\theta - 3\sin\theta + 1 = 0 \qquad 0 \le \theta < 2\pi$

$(2\sin\theta - 1)(\sin\theta - 1) = 0$

$2\sin\theta - 1 = 0$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\sin\theta - 1 = 0$

$\sin\theta = 1$

$\theta = \frac{\pi}{2}$

Example 2 Solve the equation: (Using Trig Identities)

$3\cos\theta + 3 = 2\sin^2\theta \qquad 0 \le \theta < 2\pi$

$3\cos\theta + 3 = 2(1 - \cos^2\theta)$

$3\cos\theta + 3 = 2 - 2\cos^2\theta$

$2\cos^2\theta + 3\cos\theta + 1 = 0$

$(2\cos\theta + 1)(\cos\theta + 1) = 0$

$2\cos\theta + 1 = 0$

$\cos\theta = -\frac{1}{2}$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

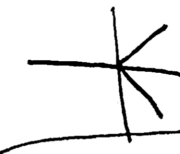
$\cos\theta + 1 = 0$

$\cos\theta = -1$

$\theta = \pi$

Example 3 Solve the equation: (Using Trig Identities)

$$\cos(2\theta)+3=5\cos\theta \qquad 0\le\theta<2\pi$$



$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$(2\cos^2\theta-1)+3=5\cos\theta$$

$$2\cos^2\theta+2-5\cos\theta=0$$

$$2\cos^2\theta-5\cos\theta+2=0$$

$$(2\cos\theta-1)(\cos\theta-2)=0$$

$$\cos\theta=\frac{1}{2} \quad\Big|\quad \cos\theta=2 \text{ no soln}$$

Example 4 Solve the equation: (Using Trig Identities)

$$\cos^2\theta+\sin\theta=2 \qquad 0\le\theta<2\pi$$

$$1-\sin^2\theta+\sin\theta=2$$

$$-1+\sin^2\theta-\sin\theta=-2$$

$$(1)\sin^2\theta-1\sin\theta+(1)=0$$

no soln

$$ax^2+bx+c=0$$

$$x^2-x+1=0$$

$$(x \quad 1)(x \quad 1)=0$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$a=1$
$b=-1$
$c=1$

$$\sqrt{(-1)^2-4(1)(1)}$$

$$\sqrt{1-4}=\sqrt{-3}$$

Example 5 Solve the equation: (Using Trig Identities)

$$\sin\theta\cos\theta = \frac{-1}{2} \qquad 0 \le \theta < 2\pi$$

$$\sin 2(\theta) = 2\sin\theta\cos\theta$$

$$(2)\,\sin\theta\cos\theta = \frac{-1}{2}\,(2)$$

$$2\sin\theta\cos\theta = -1$$

$$\sin(2\theta) = -1$$

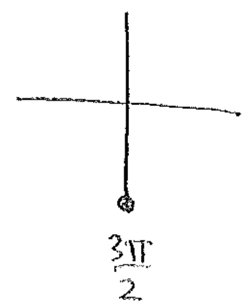
$$\frac{2\theta}{2} = \frac{\frac{3\pi}{2} + 2n\pi}{2}$$

$$\theta = \frac{3\pi}{4} + n\pi$$

Temp solve

$$\sin(\theta) = -1$$

$$\theta = \frac{3\pi}{2} + 2n\pi$$



$n = 0$

$$\theta = \frac{3\pi}{4} + 0\pi$$

$$= \frac{3\pi}{4}$$

$n = 1$

$$\theta = \frac{3\pi}{4} + 1\pi$$

$$\theta = \frac{3\pi}{4} + \frac{4\pi}{4}$$

$$\theta = \frac{7\pi}{4}$$

$n = 2$

$$\theta = \frac{3\pi}{4} + 2\pi$$

$$\theta = \frac{3\pi}{4} + \frac{8\pi}{4} = \cancel{\frac{11\pi}{4}}$$

70. **Bending Light** The speed of yellow sodium light (wavelength of 589 nanometers) in a certain liquid is measured to be 1.92×10^8 meters per second. What is the index of refraction of this liquid, with respect to air, for sodium light?*
[**Hint:** The speed of light in air is approximately 2.998×10^8 meters per second.]

71. **Bending Light** A beam of light with a wavelength of 589 nanometers traveling in air makes an angle of incidence of 40° on a slab of transparent material, and the refracted beam makes an angle of refraction of 26°. Find the index of refraction of the material.*

72. **Bending Light** A light ray with a wavelength of 589 nanometers (produced by a sodium lamp) traveling through air makes an angle of incidence of 30° on a smooth, flat slab of crown glass. Find the angle of refraction.*

73. A light beam passes through a thick slab of material whose index of refraction is n_2. Show that the emerging beam is parallel to the incident beam.*

74. **Brewster's Law** If the angle of incidence and the angle of refraction are complementary angles, the angle of incidence is referred to as the Brewster angle θ_B. The Brewster angle is related to the index of refractions of the two media, n_1 and n_2, by the equation $n_1 \sin\theta_B = n_2 \cos\theta_B$, where n_1 is the index of refraction of the incident medium and n_2 is the index of refraction of the refractive medium. Determine the Brewster angle for a light beam traveling through water (at 20°C) that makes an angle of incidence with a smooth, flat slab of crown glass.

* Adapted from Halliday and Resnick, *Fundamentals of Physics*, 7th ed., 2005, John Wiley & Sons.

Discussion and Writing

75. Explain in your own words how you would use your calculator to solve the equation $\cos x = -0.6, 0 \le x < 2\pi$. How would you modify your approach to solve the equation $\cot x = 5, 0 < x < 2\pi$?

'Are You Prepared?' Answers

1. $\left\{\frac{3}{2}\right\}$ 2. $\frac{\sqrt{2}}{2}; -\frac{1}{2}$

7.8 Trigonometric Equations (II)

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Solving Quadratic Equations by Factoring (Appendix A, Section A.6, pp. A46–A47)
- The Quadratic Formula (Appendix A, Section A.6, pp. A49–A51)
- Using a Graphing Utility to Solve Equations (Appendix B, Section B.4, pp. B8–B10)

Now Work the 'Are You Prepared?' problems on page 487.

OBJECTIVES 1 Solve Trigonometric Equations Quadratic in Form (p. 482)
2 Solve Trigonometric Equations Using Identities (p. 483)
3 Solve Trigonometric Equations Linear in Sine and Cosine (p. 485)
4 Solve Trigonometric Equations Using a Graphing Utility (p. 487)

1 Solve Trigonometric Equations Quadratic in Form

In this section we continue our study of trigonometric equations. Many trigonometric equations can be solved by applying techniques that we already know, such as applying the quadratic formula (if the equation is a second-degree polynomial) or factoring.

EXAMPLE 1 Solving a Trigonometric Equation Quadratic in Form

Solve the equation: $2\sin^2\theta - 3\sin\theta + 1 = 0, \quad 0 \le \theta < 2\pi$

Solution This equation is a quadratic equation (in $\sin\theta$) that can be factored.

$$2\sin^2\theta - 3\sin\theta + 1 = 0 \qquad 2x^2 - 3x + 1 = 0, \quad x = \sin\theta$$
$$(2\sin\theta - 1)(\sin\theta - 1) = 0 \qquad (2x - 1)(x - 1) = 0$$

$$2\sin\theta - 1 = 0 \quad \text{or} \quad \sin\theta - 1 = 0 \qquad \text{Use the Zero-Product Property.}$$
$$\sin\theta = \frac{1}{2} \quad \text{or} \quad \sin\theta = 1$$

Solving each equation in the interval $[0, 2\pi)$, we obtain

$$\theta = \frac{\pi}{6}, \quad \theta = \frac{5\pi}{6}, \quad \theta = \frac{\pi}{2}$$

The solution set is $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}\right\}$.

Now Work PROBLEM 7

2 Solve Trigonometric Equations Using Identities

When a trigonometric equation contains more than one trigonometric function, identities sometimes can be used to obtain an equivalent equation that contains only one trigonometric function.

EXAMPLE 2 Solving a Trigonometric Equation Using Identities

Solve the equation: $3\cos\theta + 3 = 2\sin^2\theta, \quad 0 \le \theta < 2\pi$

Solution The equation in its present form contains a sine and a cosine. However, a form of the Pythagorean Identity can be used to transform the equation into an equivalent expression containing only cosines.

$$\begin{aligned}
3\cos\theta + 3 &= 2\sin^2\theta \\
3\cos\theta + 3 &= 2(1 - \cos^2\theta) && \sin^2\theta = 1 - \cos^2\theta \\
3\cos\theta + 3 &= 2 - 2\cos^2\theta \\
2\cos^2\theta + 3\cos\theta + 1 &= 0 && \text{Quadratic in } \cos\theta \\
(2\cos\theta + 1)(\cos\theta + 1) &= 0 && \text{Factor.}
\end{aligned}$$
$$2\cos\theta + 1 = 0 \quad \text{or} \quad \cos\theta + 1 = 0 \qquad \text{Use the Zero-Product Property.}$$
$$\cos\theta = -\frac{1}{2} \quad \text{or} \quad \cos\theta = -1$$

Solving each equation in the interval $[0, 2\pi)$, we obtain

$$\theta = \frac{2\pi}{3}, \quad \theta = \frac{4\pi}{3}, \quad \theta = \pi$$

The solution set is $\left\{\frac{2\pi}{3}, \pi, \frac{4\pi}{3}\right\}$.

Check: Graph $Y_1 = 3\cos x + 3$ and $Y_2 = 2\sin^2 x, 0 \le x \le 2\pi$, and find the points of intersection. How close are your approximate solutions to the exact ones found in this example?

EXAMPLE 3 Solving a Trigonometric Equation Using Identities

Solve the equation: $\cos(2\theta) + 3 = 5\cos\theta, \quad 0 \le \theta < 2\pi$

Solution First, we observe that the given equation contains two cosine functions, but with different arguments, θ and 2θ. We use the Double-angle Formula $\cos(2\theta) = 2\cos^2\theta - 1$ to obtain an equivalent equation containing only $\cos\theta$.

$$\begin{aligned}
\cos(2\theta) + 3 &= 5\cos\theta \\
(2\cos^2\theta - 1) + 3 &= 5\cos\theta && \cos(2\theta) = 2\cos^2\theta - 1
\end{aligned}$$

$$\begin{aligned} 2\cos^2\theta - 5\cos\theta + 2 &= 0 && \text{Place in standard form.}\\ (\cos\theta - 2)(2\cos\theta - 1) &= 0 && \text{Factor.}\\ \cos\theta = 2 \quad \text{or} \quad \cos\theta &= \frac{1}{2} && \text{Solve by using the Zero-Product Property.} \end{aligned}$$

For any angle θ, $-1 \le \cos\theta \le 1$; therefore, the equation $\cos\theta = 2$ has no solution.

The solutions of $\cos\theta = \frac{1}{2}$, $0 \le \theta < 2\pi$, are

$$\theta = \frac{\pi}{3}, \quad \theta = \frac{5\pi}{3}$$

The solution set is $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$.

Now Work PROBLEM 23

EXAMPLE 4 **Solving a Trigonometric Equation Using Identities**

Solve the equation: $\cos^2\theta + \sin\theta = 2, \quad 0 \le \theta < 2\pi$

Solution This equation involves two trigonometric functions, sine and cosine. We use a form of the Pythagorean Identity, $\sin^2\theta + \cos^2\theta = 1$, to rewrite the equation in terms of $\sin\theta$.

$$\begin{aligned} \cos^2\theta + \sin\theta &= 2\\ (1 - \sin^2\theta) + \sin\theta &= 2 && \cos^2\theta = 1 - \sin^2\theta\\ \sin^2\theta - \sin\theta + 1 &= 0 \end{aligned}$$

This is a quadratic equation in $\sin\theta$. The discriminant is $b^2 - 4ac = 1 - 4 = -3 < 0$. Therefore, the equation has no real solution. The solution set is the empty set, $\varnothing$.

Check: Graph $Y_1 = \cos^2 x + \sin x$ and $Y_2 = 2$ to see that the two graphs never intersect, so the equation $Y_1 = Y_2$ has no real solution.

EXAMPLE 5 **Solving a Trigonometric Equation Using Identities**

Solve the equation: $\sin\theta\cos\theta = -\frac{1}{2}, \quad 0 \le \theta < 2\pi$

Solution The left side of the given equation is in the form of the Double-angle Formula $2\sin\theta\cos\theta = \sin(2\theta)$, except for a factor of 2. We multiply each side by 2.

$$\begin{aligned} \sin\theta\cos\theta &= -\frac{1}{2}\\ 2\sin\theta\cos\theta &= -1 && \text{Multiply each side by 2.}\\ \sin(2\theta) &= -1 && \text{Double-angle Formula} \end{aligned}$$

The argument here is 2θ. So we need to write all the solutions of this equation and then list those that are in the interval $[0, 2\pi)$. Because $\sin\left(\frac{3\pi}{2} + 2\pi k\right) = -1$, for any integer k we have

$$\begin{aligned} 2\theta &= \frac{3\pi}{2} + 2k\pi && k \text{ any integer}\\ \theta &= \frac{3\pi}{4} + k\pi \end{aligned}$$

$$\underset{k=-1}{\theta = \frac{3\pi}{4} + (-1)\pi = -\frac{\pi}{4}}, \quad \underset{k=0}{\theta = \frac{3\pi}{4} + (0)\pi = \frac{3\pi}{4}}, \quad \underset{k=1}{\theta = \frac{3\pi}{4} + (1)\pi = \frac{7\pi}{4}}, \quad \underset{k=2}{\theta = \frac{3\pi}{4} + (2)\pi = \frac{11\pi}{4}}$$

The solutions in the interval $[0, 2\pi)$ are

$$\theta = \frac{3\pi}{4}, \quad \theta = \frac{7\pi}{4}$$

The solution set is $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$.

3 Solve Trigonometric Equations Linear in Sine and Cosine

Sometimes it is necessary to square both sides of an equation to obtain expressions that allow the use of identities. Remember, squaring both sides of an equation may introduce extraneous solutions. As a result, apparent solutions must be checked.

EXAMPLE 6 **Solving a Trigonometric Equation Linear in Sine and Cosine**

Solve the equation: $\sin\theta + \cos\theta = 1, \quad 0 \le \theta < 2\pi$

Solution A Attempts to use available identities do not lead to equations that are easy to solve. (Try it yourself.) Given the form of this equation, we decide to square each side.

$$\begin{aligned}
\sin\theta + \cos\theta &= 1 \\
(\sin\theta + \cos\theta)^2 &= 1 && \text{Square each side.} \\
\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta &= 1 && \text{Remove parentheses.} \\
2\sin\theta\cos\theta &= 0 && \sin^2\theta + \cos^2\theta = 1 \\
\sin\theta\cos\theta &= 0
\end{aligned}$$

Setting each factor equal to zero, we obtain

$$\sin\theta = 0 \quad \text{or} \quad \cos\theta = 0$$

The apparent solutions are

$$\theta = 0, \quad \theta = \pi, \quad \theta = \frac{\pi}{2}, \quad \theta = \frac{3\pi}{2}$$

Because we squared both sides of the original equation, we must check these apparent solutions to see if any are extraneous.

$$\begin{aligned}
&\theta = 0: && \sin 0 + \cos 0 = 0 + 1 = 1 && \text{A solution} \\
&\theta = \pi: && \sin\pi + \cos\pi = 0 + (-1) = -1 && \text{Not a solution} \\
&\theta = \frac{\pi}{2}: && \sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 1 + 0 = 1 && \text{A solution} \\
&\theta = \frac{3\pi}{2}: && \sin\frac{3\pi}{2} + \cos\frac{3\pi}{2} = -1 + 0 = -1 && \text{Not a solution}
\end{aligned}$$

The values $\theta = \pi$ and $\theta = \frac{3\pi}{2}$ are extraneous. The solution set is $\left\{0, \frac{\pi}{2}\right\}$.

Solution B We start with the equation

$$\sin\theta + \cos\theta = 1$$

and divide each side by $\sqrt{2}$. (The reason for this choice will become apparent shortly.) Then

$$\frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta = \frac{1}{\sqrt{2}}$$

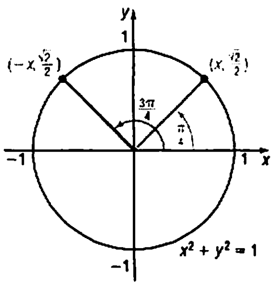
The left side now resembles the formula for the sine of the sum of two angles, one of which is θ. The other angle is unknown (call it ϕ.) Then

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \tag{1}$$

where

$$\cos\phi = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \qquad \sin\phi = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \qquad 0 \le \phi < 2\pi$$

Figure 30



The angle ϕ is therefore $\frac{\pi}{4}$. As a result, equation (1) becomes

$$\sin\left(\theta + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

In the interval $[0, 2\pi)$, there are two angles whose sine is $\frac{\sqrt{2}}{2}$: $\frac{\pi}{4}$ and $\frac{3\pi}{4}$. See Figure 30. As a result,

$$\theta + \frac{\pi}{4} = \frac{\pi}{4} \quad \text{or} \quad \theta + \frac{\pi}{4} = \frac{3\pi}{4}$$
$$\theta = 0 \quad \text{or} \quad \theta = \frac{\pi}{2}$$

The solution set is $\left\{0, \frac{\pi}{2}\right\}$.

This second method of solution can be used to solve any linear equation in the variables $\sin\theta$ and $\cos\theta$. Let's look at an example.

EXAMPLE 7 Solving a Trigonometric Equation Linear in Sin θ and Cos θ

Solve:

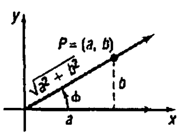
$$a\sin\theta + b\cos\theta = c \tag{2}$$

where a, b, and c are constants and either $a \ne 0$ or $b \ne 0$.

Solution We divide each side of equation (2) by $\sqrt{a^2 + b^2}$. Then

$$\frac{a}{\sqrt{a^2 + b^2}}\sin\theta + \frac{b}{\sqrt{a^2 + b^2}}\cos\theta = \frac{c}{\sqrt{a^2 + b^2}} \tag{3}$$

Figure 31



There is a unique angle ϕ, $0 \le \phi < 2\pi$, for which

$$\cos\phi = \frac{a}{\sqrt{a^2 + b^2}} \quad \text{and} \quad \sin\phi = \frac{b}{\sqrt{a^2 + b^2}} \tag{4}$$

Figure 31 shows the situation for $a > 0$ and $b > 0$. Equation (3) may be written as

$$\sin\theta\cos\phi + \cos\theta\sin\phi = \frac{c}{\sqrt{a^2 + b^2}}$$

or, equivalently,

$$\sin(\theta + \phi) = \frac{c}{\sqrt{a^2 + b^2}} \tag{5}$$

where ϕ satisfies equation (4).

If $|c| > \sqrt{a^2 + b^2}$, then $\sin(\theta + \phi) > 1$ or $\sin(\theta + \phi) < -1$, and equation (5) has no solution.

If $|c| \le \sqrt{a^2 + b^2}$, then all the solutions of equation (5) are

$$\theta + \phi = \sin^{-1}\frac{c}{\sqrt{a^2+b^2}} + 2k\pi \quad \text{or} \quad \theta + \phi = \pi - \sin^{-1}\frac{c}{\sqrt{a^2+b^2}} + 2k\pi$$

Because the angle ϕ is determined by equations (4), these provide the solutions to equation (2).

Now Work PROBLEM 41

4 Solve Trigonometric Equations Using a Graphing Utility

The techniques introduced in this section apply only to certain types of trigonometric equations. Solutions for other types are usually studied in calculus, using numerical methods. In the next example, we show how a graphing utility may be used to obtain solutions.

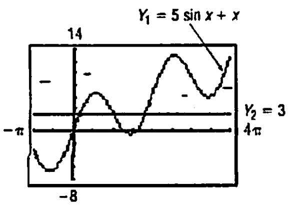
EXAMPLE 8 Solving Trigonometric Equations Using a Graphing Utility

Solve: $5 \sin x + x = 3$

Express the solution(s) rounded to two decimal places.

Solution This type of trigonometric equation cannot be solved by previous methods. A graphing utility, though, can be used here. The solution(s) of this equation is the same as the points of intersection of the graphs of $Y_1 = 5 \sin x + x$ and $Y_2 = 3$. See Figure 32.

Figure 32



There are three points of intersection; the x-coordinates are the solutions that we seek. Using INTERSECT, we find

$$x = 0.52, \quad x = 3.18, \quad x = 5.71$$

The solution set is $\{0.52, 3.18, 5.71\}$.

Now Work PROBLEM 53

7.8 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Find the real solutions of $4x^2 - x - 5 = 0$. (pp. A46–A47)
2. Find the real solutions of $x^2 - x - 1 = 0$. (pp. A49–A51)
3. Find the real solutions of $(2x - 1)^2 - 3(2x - 1) - 4 = 0$. (pp. A49–A51)
4. Use a graphing utility to solve $5x^3 - 2 = x - x^2$. Round answers to two decimal places. (pp. B8–B10)

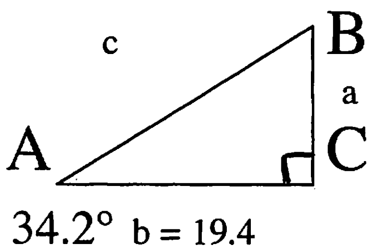
Skill Building

In Problems 5–46, solve each equation on the interval $0 \le \theta < 2\pi$.

5. $2\cos^2\theta + \cos\theta = 0$
6. $\sin^2\theta - 1 = 0$
7. $2\sin^2\theta - \sin\theta - 1 = 0$
8. $2\cos^2\theta + \cos\theta - 1 = 0$
9. $(\tan\theta - 1)(\sec\theta - 1) = 0$
10. $(\cot\theta + 1)\left(\csc\theta - \frac{1}{2}\right) = 0$
11. $\sin^2\theta - \cos^2\theta = 1 + \cos\theta$
12. $\cos^2\theta - \sin^2\theta + \sin\theta = 0$
13. $\sin^2\theta = 6(\cos\theta + 1)$
14. $2\sin^2\theta = 3(1 - \cos\theta)$
15. $\cos(2\theta) + 6\sin^2\theta = 4$
16. $\cos(2\theta) = 2 - 2\sin^2\theta$
17. $\cos\theta = \sin\theta$
18. $\cos\theta + \sin\theta = 0$
19. $\tan\theta = 2\sin\theta$
20. $\sin(2\theta) = \cos\theta$
21. $\sin\theta = \csc\theta$
22. $\tan\theta = \cot\theta$
23. $\cos(2\theta) = \cos\theta$
24. $\sin(2\theta)\sin\theta = \cos\theta$
25. $\sin(2\theta) + \sin(4\theta) = 0$
26. $\cos(2\theta) + \cos(4\theta) = 0$
27. $\cos(4\theta) - \cos(6\theta) = 0$
28. $\sin(4\theta) - \sin(6\theta) = 0$

§8.1 Applications Involving Right Triangles

Example 1 Solving a right triangle. Find all angles and sides of the following right triangle.



$a^2 + b^2 = c^2$ $\quad (23.456)^2 = a^2 + (19.4)^2$

Sohcahtoa $\quad a^2 = 173.823$

$C = 90°$ $\quad b = 19.4$

$A = 34.2°$ $\quad c = 23.456$

$a = 13.184$

$B = ?\ 180° - 34.2° - 90°$

$= \boxed{55.8°}$

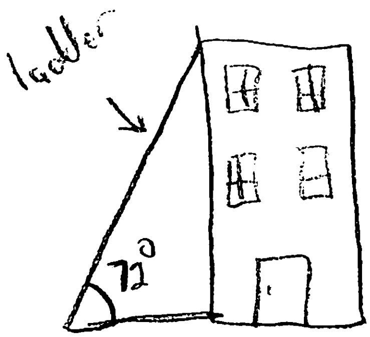
$\cos A = \frac{adj}{hyp}$

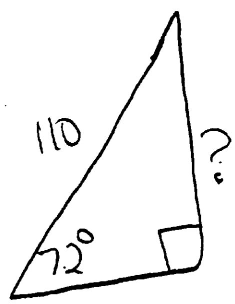
$\frac{\cos 34.2°}{1} = \frac{19.4}{c}$

$c(\cos 34.2°) = 19.4$

$c = \frac{19.4}{\cos 34.2°} = 23.456$

Example 2 A safety regulation states that the maximum angle of elevation for a rescue ladder is 72°. A fire department's longest ladder is 110 feet. What is the maximum safe rescue height?



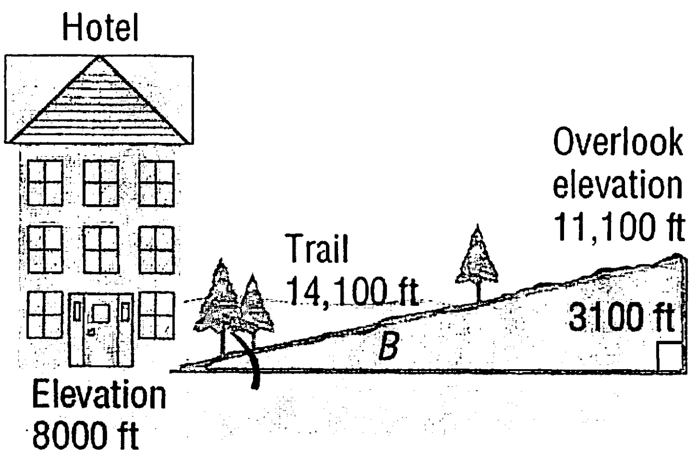


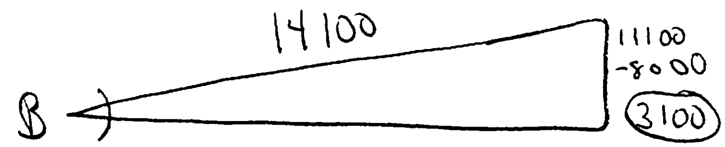
$\frac{\sin 72°}{1} = \frac{x}{110}$

$x = 110(\sin 72°)$

$x = 104.616$

Example 3 A straight trail leads from the Alpine Hotel, elevation 8000 feet, to a scenic overlook, elevation 11,100 feet. The length of the trail is 14,100 feet. What is the inclination (grade) of the trail? That is, what is the angle B in the figure?

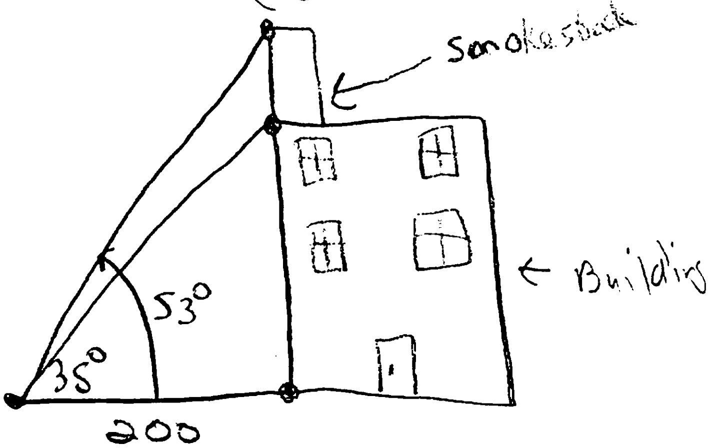


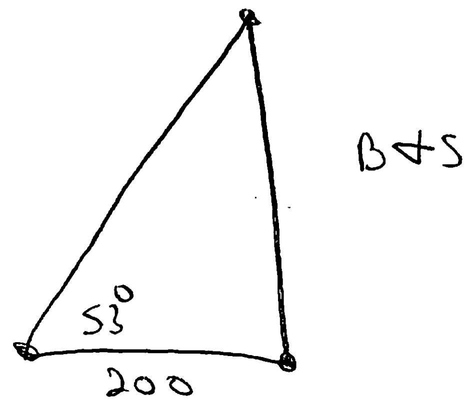


$$\sin B = \frac{3100}{14100}$$

$$\sin^{-1}\left(\frac{3100}{14100}\right) = 12.7^\circ$$

Example 4 At a point 200 feet from the base of a building, the angle of elevation to the bottom of a smokestack is 35°, whereas the angle of elevation to the top is 53°. Find the height s of the smoke stack alone.



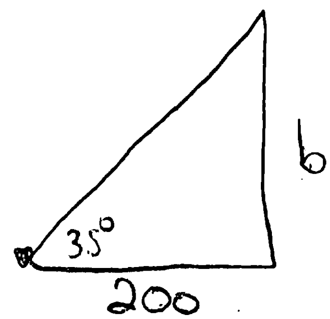


$$\tan 53^{\circ} = \frac{t}{200}$$

$$t = 200 \tan 53^{\circ}$$

$$s = 200 \tan 53^{\circ} - 200 \tan 35^{\circ}$$

$$s = 125.367$$



$$\tan 35^{\circ} = \frac{b}{200}$$

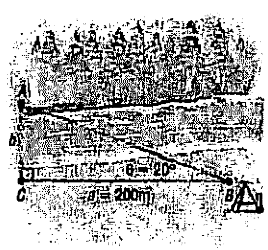
$$b = 200 \tan 35^{\circ}$$

problems, pay attention to the known measures. This will indicate what trigonometric function to use. For example, if we know the measure of an angle and the length of the side adjacent to the angle, and wish to find the length of the opposite side, we would use the tangent function. Do you know why?

EXAMPLE 6 Finding the Width of a River

A surveyor can measure the width of a river by setting up a transit* at a point C on one side of the river and taking a sighting of a point A on the other side. Refer to Figure 10. After turning through an angle of 90° at C, the surveyor walks a distance of 200 meters to point B. Using the transit at B, the angle θ is measured and found to be 20°. What is the width of the river rounded to the nearest meter?

Figure 10



Solution We seek the length of side b. We know a and θ. So we use the fact that b is opposite θ and a is adjacent to θ and write

$$\tan\theta = \frac{b}{a}$$

which leads to

$$\tan 20° = \frac{b}{200}$$

$$b = 200\tan 20° \approx 72.79 \text{ meters}$$

The width of the river is 73 meters, rounded to the nearest meter.

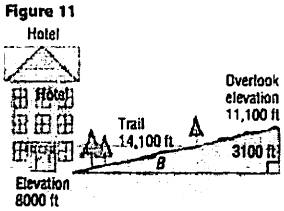
Now Work PROBLEM 49

EXAMPLE 7 Finding the Inclination of a Mountain Trail

A straight trail leads from the Alpine Hotel, elevation 8000 feet, to a scenic overlook, elevation 11,100 feet. The length of the trail is 14,100 feet. What is the inclination (grade) of the trail? That is, what is the angle B in Figure 11?

Solution As we can see in Figure 11, we know the length of the side opposite angle B and the length of the hypotenuse. The angle B obeys the equation

Figure 11



$$\sin B = \frac{3100}{14{,}100}$$

Using a calculator,

$$B = \sin^{-1}\frac{3100}{14{,}100} \approx 12.7°$$

The inclination (grade) of the trail is approximately 12.7°.

Now Work PROBLEM 55

* An instrument used in surveying to measure angles.

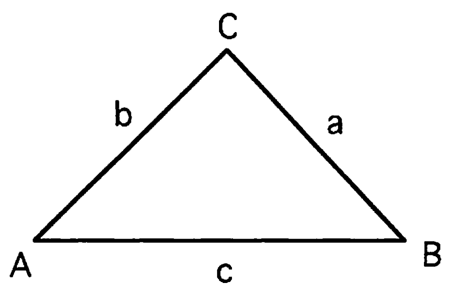
§ 8.2 Law of Sines

<u>Solving a triangle:</u> find all angles and all sides

<u>Oblique triangle</u>: triangles with no right angle. To solve an oblique triangle, you need to know the measure of at least one side and any two other parts of the triangle.

Sum of angles in a triangle: 180°

<u>The Law of Sines</u> - If A, B and C are the measures of the angles of a triangle and a, b and c are the lengths of the sides opposite these angles, then



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

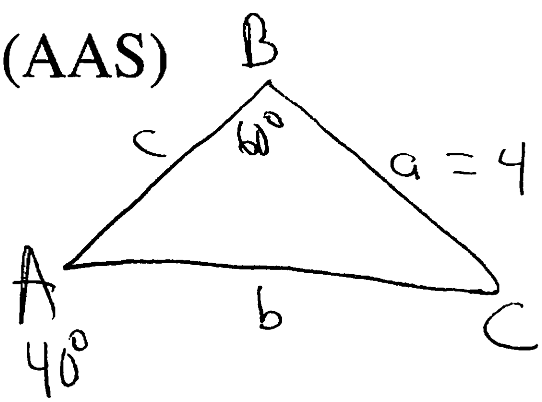
Applications of Law of Sines:

- can be used to solve an oblique triangle if two angles and a side are given (ASA or AAS)

- given two sides and an angle opposite one of the sides, the triangle may not exist or two triangles may exist or the triangle may be unique (SSA)

Example 1 Solve triangle (AAS)

A = 40°, B = 60° and a = 4



$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin 40°}{4} = \frac{\sin 60°}{b}$$

$$\frac{\sin 40° \, b}{\sin 40°} = \frac{\sin 60° \, 4}{\sin 40°}$$

$$b = 5.39$$

$a = 4$ $\quad A = 40°$

$b = 5.39$ $\quad B = 60°$

$c = 6.13$ $\quad C = 180° - 40° - 60° = 80°$

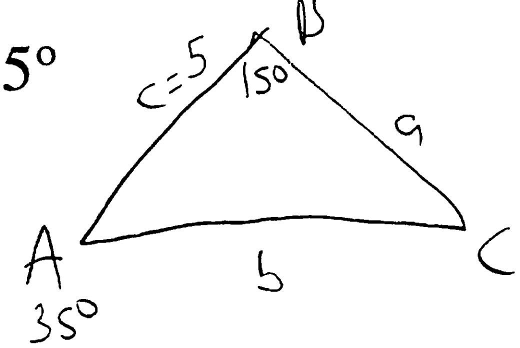
$$\frac{\sin 40°}{4} = \frac{\sin C}{c}$$

$$\frac{\sin 40°}{4} = \frac{\sin 80°}{c}$$

$$c = \frac{\sin 80° (4)}{\sin 40} = 6.13$$

Example 2 Solve triangle (ASA)

A = 35°, c = 5 and B = 15°



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin 35^\circ}{a} = \frac{\sin 130^\circ}{5}$$

$$\Rightarrow \quad a = \frac{5 \sin 35^\circ}{\sin 130^\circ}$$

$$= 3.74$$

$a = 3.74$ $A = 35^\circ$

$b =$ $B = 15^\circ$

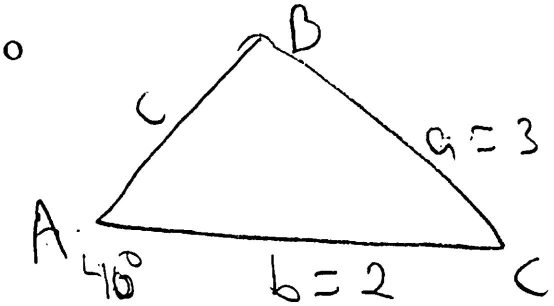
$c = 5$ $C = 180 - 35^\circ - 15^\circ = 130^\circ$

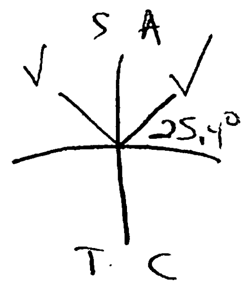
$$\frac{\sin 130^\circ}{5} = \frac{\sin 15^\circ}{b}$$

$$b = \frac{5 \sin 15^\circ}{\sin 130^\circ} = 1.69$$

Example 3 Solve triangle (SSA) one solution.

a = 3, b = 2 and A = 40°





$$\frac{\sin A}{a} = \frac{\sin B^\circ}{b}$$

$$\frac{\sin 40^\circ}{3} = \frac{\sin B^\circ}{2}$$

$$\frac{2 \sin 40^\circ}{3} = \frac{3 \sin B^\circ}{3}$$

$$\sin B^\circ = 0.43$$

$$B^\circ = \sin^{-1}(.43) = 25.4^\circ$$

$B = 25.4^\circ$ $b = 2$

$A = 40^\circ$ $a = 3$

$C = 114.6^\circ$ $c = 4.24$

$180 - 25.4^\circ$

$= \cancel{154.6^\circ}$

$B_1 = 25.4^\circ$

$\cancel{B_2 = 154.6^\circ}$

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

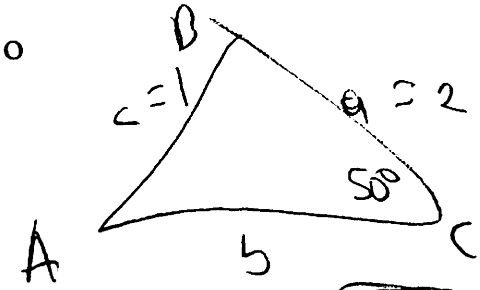
$$\frac{\sin 114.6}{c} = \frac{\sin 40^\circ}{3}$$

$$c = \frac{3 \sin 114.6^\circ}{\sin 40^\circ}$$

$$= 4.24$$

Example 4 Solve triangle (SSA) No solution.

a = 2, c = 1 and C = 50°



$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\frac{\sin 50^\circ}{1} = \frac{\sin A}{2}$$

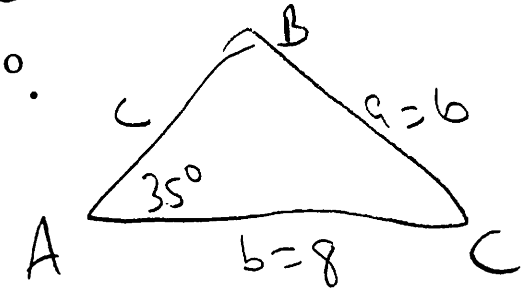
no soln

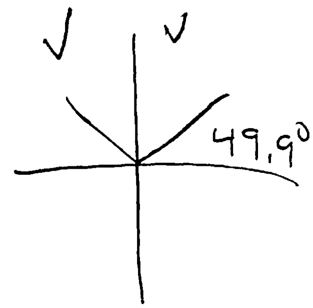
$$\sin A^\circ = 2\sin 50^\circ$$

$$A^\circ = \sin^{-1}(2\sin 50^\circ) = \sin^{-1}(1.53)$$

Example 5 Solve triangle (SSA) two solutions

a = 6, b = 8 and A = 35°.





√ $B_1 = 49.9^\circ$

√ $B_2 = 130.1^\circ$

$$\frac{\sin A}{a} = \frac{\sin B^\circ}{b}$$

$$\frac{\sin 35^\circ}{6} = \frac{\sin B^\circ}{8}$$

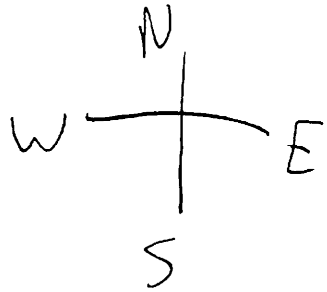
$$\sin B^\circ = \frac{8\sin 35^\circ}{6} =$$

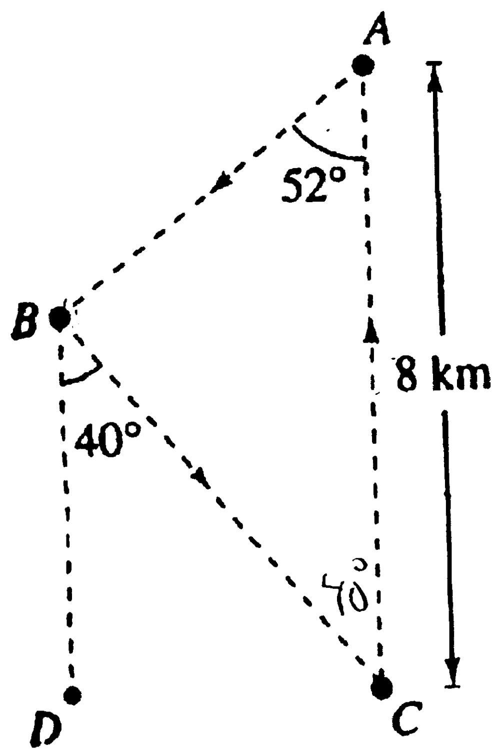
$$B^\circ = \sin^{-1}(.76)$$

$$B^\circ = 49.9^\circ$$

triangle 1		triangle 2
b=8	$B_1 = 49.9^\circ$	b = 8 $B_2 = 130.1^\circ$
a=6	A = 35°	a = 6 A = 35°
c=	C = 95.1°	c = C = 14.9°
	c = 10.4	C = 2.7

Example 6 The course for a boat race starts at point A and proceeds in the direction S 52° W to point B, then in the direction S 40° E to point C, and finally back to A, as shown in figure. Point C lies 8 kilometers directly south of point A. Approximate the total distance of the race course.





$b = 8$ $\qquad$ $B = 88^\circ$

$a =$ $\qquad$ $A = 52^\circ$

$c =$ $\qquad$ $C = 40^\circ$

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\frac{\sin 88^\circ}{8} = \frac{\sin 52^\circ}{a}$$

$$\frac{a \sin 88^\circ}{\sin 88^\circ} = \frac{8 \sin 52^\circ}{\sin 88^\circ}$$

$a \approx 6.308$

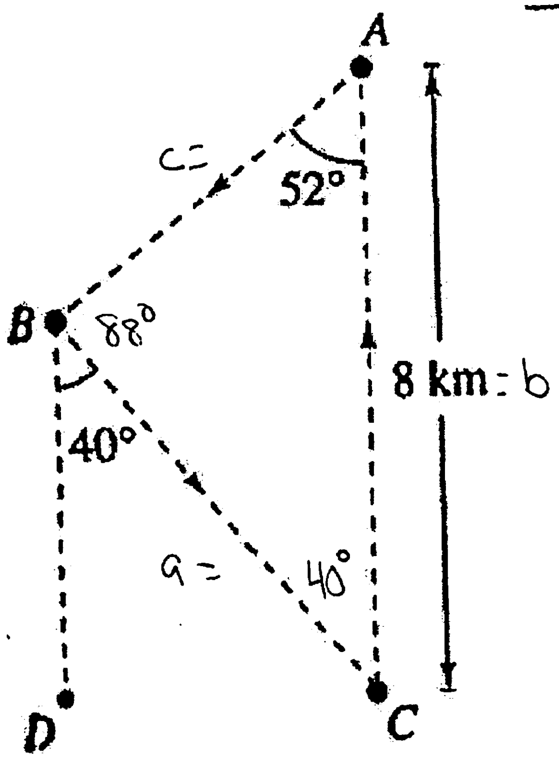
19.453

$$\frac{\sin 40^\circ}{c} = \frac{\sin 88^\circ}{8}$$

$$c = \frac{8 \sin 40^\circ}{\sin 88^\circ}$$

$= 5.145$

Example 7 The course for a boat race starts at point A and proceeds in the direction S 52° W to point B, then in the direction S 40° E to point C, and finally back to A, as shown in figure 6.9 (textbook page 415). Point C lies 8 kilometers directly south of point A. Approximate the total distance of the race course.



$B = 180° - 52° - 40° = 88°$

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

$$= \frac{c}{\sin 40°} = \frac{8}{\sin 88°}$$

$$= c = \frac{8 \sin 40°}{\sin 88°} = \boxed{5.145}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin 52°} = \frac{8}{\sin 88°}$$

$$a = \frac{8 \sin 52°}{\sin 88°} = \boxed{6.308}$$

add all sides

$$a + b + c = \boxed{19.453 \text{ km}}$$

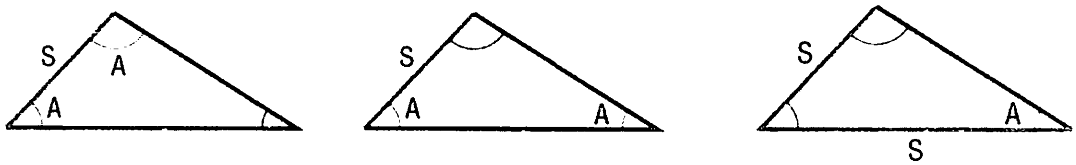
The **Law of Sines** is used to solve triangles for which Case 1 or 2 holds.

Theorem

Law of Sines

For a triangle with sides a, b, c and opposite angles A, B, C, respectively,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{1}$$



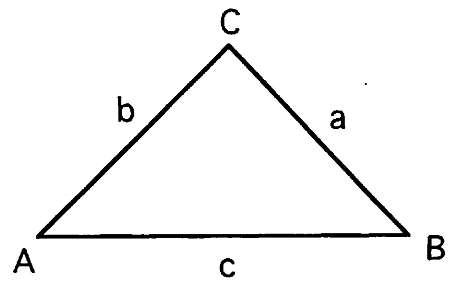
Case 1: ASA   Case 1: SAA   Case 2: SSA

$$\frac{\sin A}{a} = \frac{\sin B}{b} \qquad \frac{\sin A}{a} = \frac{\sin C}{c} \qquad \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$A + B + C = 180°$$

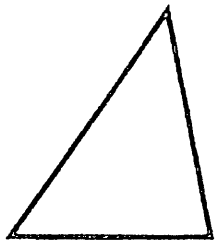
§ 8.3 Law of Cosines

The Law of Cosines is used to solve triangles in which two sides and the included angle (the angle between the two sides) are known or in which three sides are known (SAS or SSS)

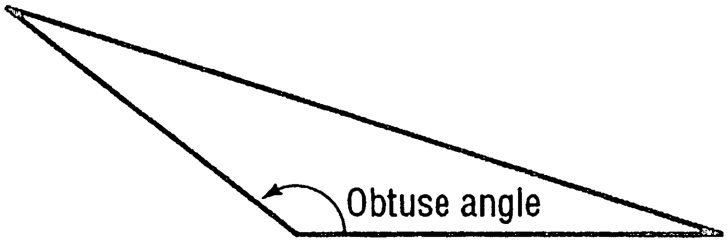


<u>The Law of Cosines</u> - If A, B and C are the measures of the angles of a triangle and a, b and c are the lengths of the sides opposite these angles, then
$a^2 = b^2 + c^2 - 2bc\cos A$ or $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$
$b^2 = a^2 + c^2 - 2ac\cos B$ or $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$
$c^2 = a^2 + b^2 - 2ab\cos C$ or $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Note: It is wise to find the largest angle which is across the largest side FIRST !

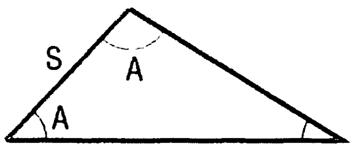


(a) All angles are acute

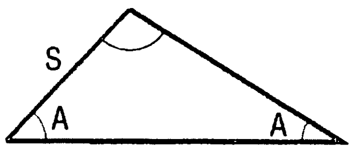


(b) Two acute angles and one obtuse angle

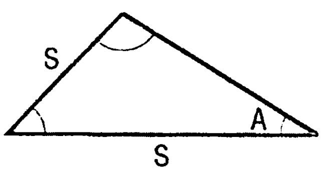
CASE 1: One side and two angles are known (ASA or SAA).
CASE 2: Two sides and the angle opposite one of them are known (SSA).
CASE 3: Two sides and the included angle are known (SAS).
CASE 4: Three sides are known (SSS).



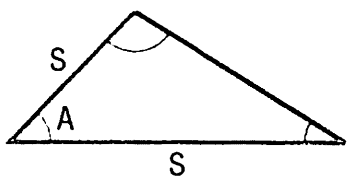
Case 1: ASA



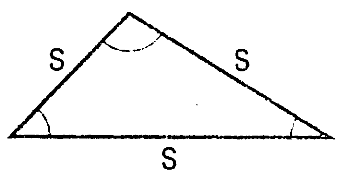
Case 1: SAA



Case 2: SSA



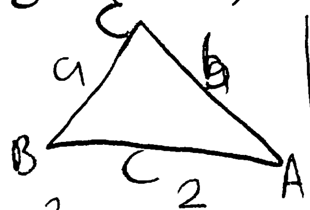
Case 3: SAS



Case 4: SSS

Example 1 Solve triangle (SSS)

a = 4, b = 3, and c = 6.



$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

$$\cos C = \frac{4^2+3^2-6^2}{2(4)(3)}$$

$$\cos C = \frac{-11}{24} = \cos^{-1}\left(-\frac{11}{24}\right) = 117.3^\circ$$

$$\cos B = \frac{a^2+c^2-b^2}{2ac}$$

$$\cos B = \frac{16+36-9}{2(4)(6)}$$

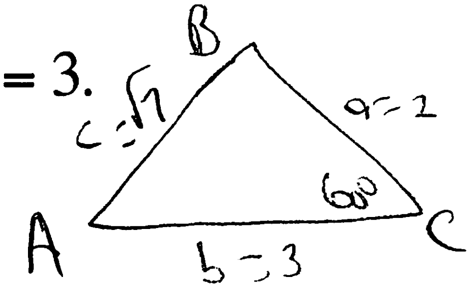
$$\cos B = \frac{43}{48}$$

$$\cos^{-1}\left(\frac{43}{48}\right) = 26.4^\circ$$

$$A = 180 - B - C = 180 - 117.3^\circ - 26.4^\circ = 36.3^\circ$$

Example 2 Solve triangle (SAS)

C = 60°, a = 2 and b = 3.



$$c^2 = a^2+b^2 - 2ab\cos C^\circ$$

$$c^2 = 2^2+3^2-2(2)(3)\cos 60^\circ$$

$$c^2 = 4+9-12\cos 60^\circ$$

$$c^2 = 13 - 12\left(\tfrac{1}{2}\right)$$

$$c^2 = 7 \qquad c = \sqrt{7} = 2.$$

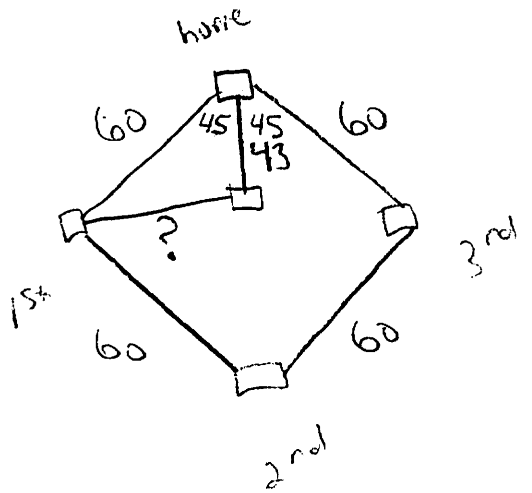
$$\cos B = \frac{a^2+c^2-b^2}{2ac}$$

$$\cos B = \frac{2^2+(\sqrt{7})^2-3^2}{2(2)(\sqrt{7})}$$

$$\cos^{-1}\left(\frac{2}{4\sqrt{7}}\right) = 79.1^\circ$$

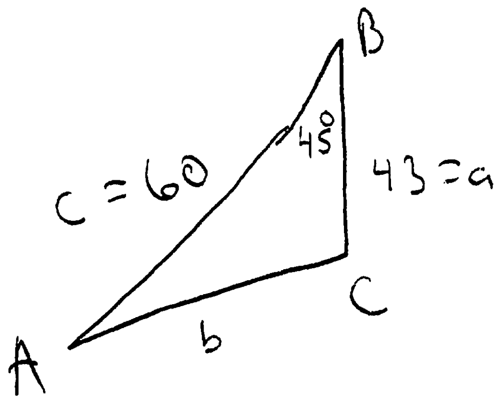
$$A = 180^\circ - 60^\circ - 79.1^\circ = 40.9^\circ$$

Example 3 The pitchers mound on a women's softball field is 43 feet from home plate and the distance between the bases is 60 feet. How far is the pitchers mound from first base?



SAS

~~SSS~~



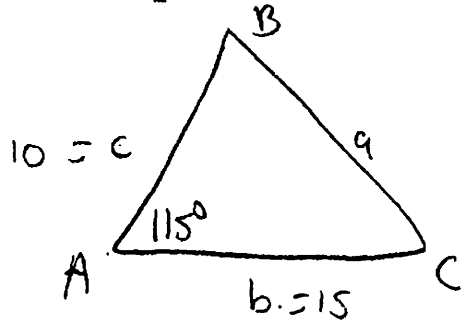
$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$b^2 = (43)^2 + (60)^2 - 2(43)(60)\cos 45^\circ$$

$$b^2 = 1800.3$$

$$b = 42.43 \text{ ft}$$

Example 2 Solve triangle (SAS) $A = 115°$, $c = 10$ and $b = 15$.



$a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 15^2 + 10^2 - 2(15)(10)\cos 115°$

$a^2 = 451.785$

$\boxed{a = 21.26}$

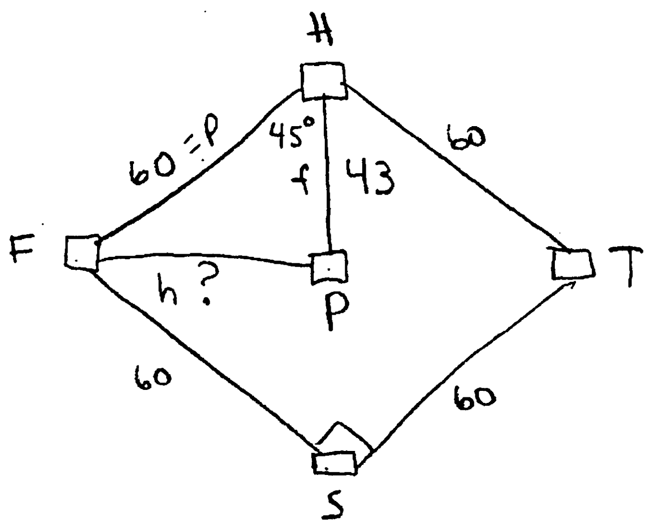
$\frac{b}{\sin B} = \frac{a}{\sin A}$

$C = 180° - 115° - 39.75°$

$\sin B = \frac{b \sin A}{a} = \frac{15 \sin 115}{21.26} = .6394$ $\boxed{C = 25.25°}$

$\sin B = .6394$ $\boxed{B = 39.75°}$

Example 3 The pitchers mound on a womens softball field is 43 feet from home plate and the distance between the bases is 60 feet. How far is the pitchers mound from first base?



ΔHPF $H = 45°$

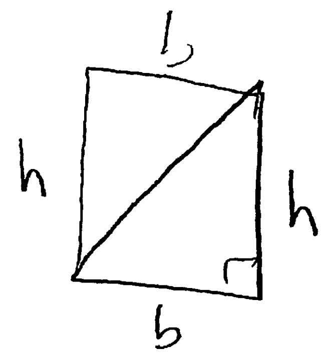
SAS

$h^2 = f^2 + p^2 - 2fp\cos H$

$h^2 = 43^2 + 60^2 - 2(43)(60)\cos 45°$

$h^2 = 1800.3$

$\boxed{h = \sqrt{1800.3} = 42.43 \text{ ft}}$



$$A = \frac{1}{2} bh$$

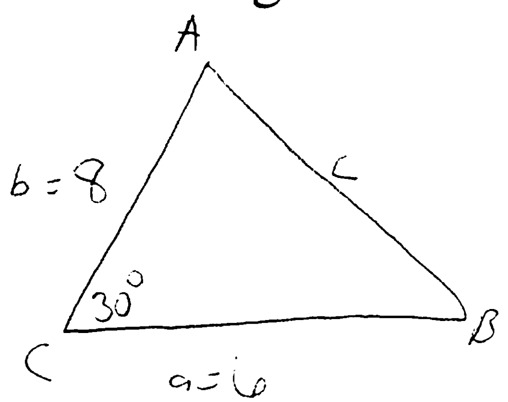
§ 8.4 Area of a Triangle

Area of a Triangle (SAS) - The area of triangle ABC is one-half the product of the lengths of any two sides and the sine of the included angle.

$$K = \frac{1}{2}bc\sin A = \frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B$$

or or

Example 1 Find the area K of a triangular lot having two sides of lengths 8 meters and 6 meters and an included angle of 30°.



use $K = \frac{1}{2} ab \sin C$

$= \frac{1}{2}(6)(8)\sin 30^\circ$

$= \left(\frac{1}{2}\right)(6)(8)\left(\frac{1}{2}\right)$

$= \boxed{12}$

Heron's Formula (SSS)- If a, b and c are the lengths of the sides of a triangle, then the area of the triangle is

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{where } s = \frac{1}{2}(a+b+c)$$

Example 2 Find the area of a triangle having sides of lengths a = 4 meters, b = 5 meters, and c = 7 meters.

$$s = \frac{1}{2}(4+5+7) = \frac{1}{2}(16) = 8$$

$$K = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{8(8-4)(8-5)(8-7)}$$

$$= \sqrt{(8)(4)(3)(1)} = \sqrt{96} = \boxed{4\sqrt{6}}$$