§5.5 Real Zeros of Polynomial Functions

Division Algorithm :

> For any polynomial P(x) and any complex number d(x) , there exists a unique polynomial Q(x) and number r(x) such that:
>
> $$P(x) = d(x) * Q(x) + r(x).$$

Example 1: Divide

a) $6q^3 - 17q^2 + 22q - 23$ by $2q - 3$

b) $3x^3 - 2x^2 - 150$ by $x - 4$

Synthetic Division :

$$\begin{array}{r} 3x^2 + 10x + 40 \\ x-4\overline{)3x^3 - 2x^2 + 0x - 150} \\ (-)\ \underline{3x^3 - 12x^2} \\ 10x^2 + 0x \\ (-)\ \underline{10x^2 - 40x} \\ 40x - 150 \\ (-)\ \underline{40x - 160} \\ 10 \end{array}$$ Answer: $3x^2 + 10x + 40 + \frac{10}{x-4}$	$$\begin{array}{r} 4\overline{)3 \quad -2 \quad 0 \quad -150} \\ \underline{12 \quad 40 \quad 160} \\ 3 \quad 10 \quad 40 \quad 10 \end{array}$$ Answer: $3x^2 + 10x + 40 + \frac{10}{x-4}$

Example 2: Divide by synthetic division.

$x^4 - 10x^2 - 2x + 4$ by $x + 3$

The Remainder Theorem

If a polynomial f(x) is divided by x – k, the remainder is equal to f(k).

Example 3: Find the remainder if $f(x) = x^3 - 4x^2 - 5$ is divided by a) x – 3 b) x + 2

The Factor Theorem

The polynomial x – k is a factor of the polynomial f(x) if and only if f(k) = 0.

Example 4: Use the Factor Theorem to determine whether the function $f(x) = 2x^3 - x^2 + 2x - 3$ has the factor

a) x – 1 b) x + 3

Number of Real Zeros (Theorem)

A polynomial function cannot have more real zeros than its degree.

Descartes' Rule of Signs

Let f denote a polynomial function written in standard form.

The number of positive real zeros of f either equals the number of variations in the sign of the nonzero coefficients of $f(x)$ or else equals that number less an even integer.

The number of negative real zeros of f either equals the number of variations in the sign of the nonzero coefficients of $f(-x)$ or else equals that number less an even integer.

Discuss the real zeros of $3x^6 - 4x^4 + 3x^3 + 2x^2 - x - 3$

Rational Zeros Theorem:

If the polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$$

has integer coefficients, every rational zero of f(x) has the form

$$\text{Rational Zero} = \frac{p}{q}$$

where p and q have no common factors other than 1, and

p is a factor of the constant term a_0 and
q is a factor of the leading coefficient a_n

$$\text{Possible rational zeros} = \frac{\text{factors of constant term } a_0}{\text{factors of leading coefficient } a_n}$$

Example 1: List the possible rational zeros for each function

a) $f(x) = 2x^3 + 3x^2 - 8x + 3$

b) $f(x) = 2x^3 + 11x^2 - 7x - 6$

Now that we have a list of possible zeros, we need to determine which possible zeros are actual zeros.

Steps for Finding the Real Zeros of a Polynomial Function

STEP 1: Use the degree of the polynomial to determine the maximum number of zeros.

STEP 2: Use Descartes' Rule of Signs to determine the possible number of positive zeros and negative zeros.

STEP 3: (a) If the polynomial has integer coefficients, use the Rational Zeros Theorem to identify those rational numbers that potentially could be zeros.

(b) Use substitution, synthetic division, or long division to test each potential rational zero.

(c) Each time that a zero (and thus a factor) is found, repeat Step 3 on the depressed equation.

STEP 4: In attempting to find the zeros, remember to use (if possible) the factoring techniques that you already know (special products, factoring by grouping, and so on).

Example : Find all the zeros for the function.

$$f(x) = x^5 - 5x^4 + 12x^3 - 24x^2 + 32x - 16$$

Intermediate Value Theorem:

Let f denote a polynomial function. If $a < b$ and if $f(a)$ and $f(b)$ are of opposite sign, there is at least one real zero of f between a and b.

example: Show that $f(x) = x^5 - x^3 - 1$ has a zero between 1 and 2.