

## Section 2.2 & 2.4 Limit of a Function (Graphically, Numerically, Analytically)

Review of Function Notation:  $f(x) = x^2 + 3x + 5$

$f$  is the name of the function

$f(x)$  indicates that  $x$  is the variable

$f(x)$  is another name for  $y$

$f(2)$  means to substitute 2 for the variable  $x$

$$f(2) = 2^2 + 3 \cdot 2 + 5 = 15$$

$f(2)$  is the  $y$ -value when the  $x$ -value is 2

The point  $(2, 15)$  is on the graph of  $f(x)$

### Limit Notation

The limit is a value that is approached, you can get closer and closer, may never reach.

$\lim_{x \rightarrow c^+} f(x)$  is the limit of  $f(x)$  as  $x$  approaches  $c$  from the right

$\lim_{x \rightarrow c^-} f(x)$  is the limit of  $f(x)$  as  $x$  approaches  $c$  from the left

$\lim_{x \rightarrow c} f(x)$  is THE limit of  $f(x)$  as  $x$  approaches  $c$

### Facts about Limits:

If the right and left limits are the same, then that number is THE limit.

If the right and left limits are not the same, then THE limit does not exist (DNE).

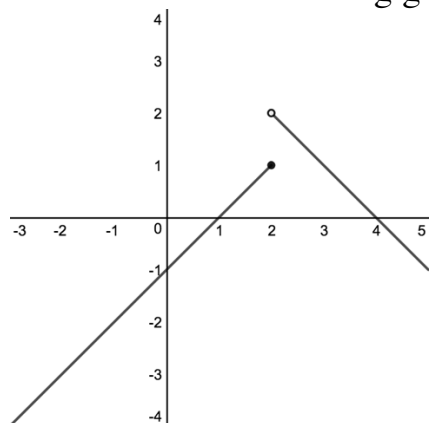
If  $f(x)$  is increasing without bound, the limit is  $+\infty$ .

If  $f(x)$  is decreasing without bound, the limit is  $-\infty$ .

Limits can also be used to describe behavior at the extreme right and left sides of the graph.

$\lim_{x \rightarrow -\infty} f(x)$  and  $\lim_{x \rightarrow +\infty} f(x)$  refer to end behavior

Let's examine limits using graphs.



### **Example 1:**

Find the following using the graph on the left.

$f(2)$

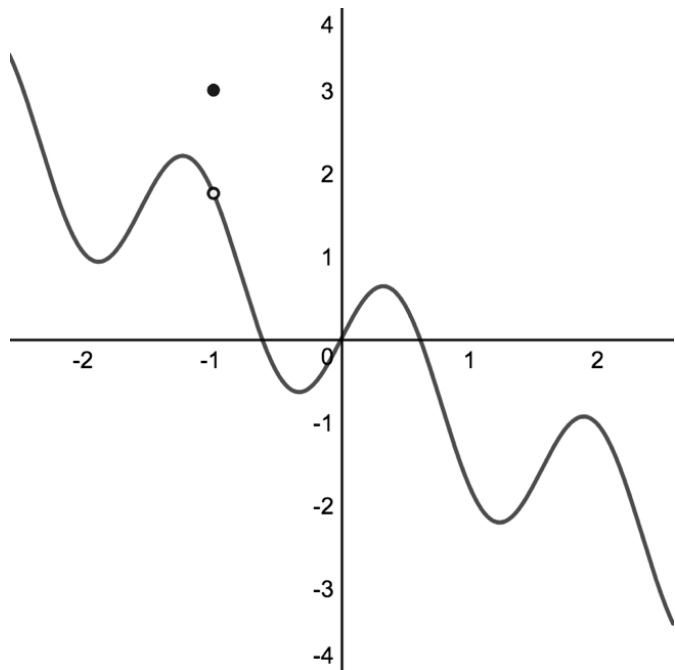
$\lim_{x \rightarrow 2^+} f(x)$

$\lim_{x \rightarrow 2^-} f(x)$

$\lim_{x \rightarrow 2} f(x)$

$\lim_{x \rightarrow -\infty} f(x)$

$\lim_{x \rightarrow +\infty} f(x)$



**Example 2:**

Find the following using the graph on the left.

$$f(-1)$$

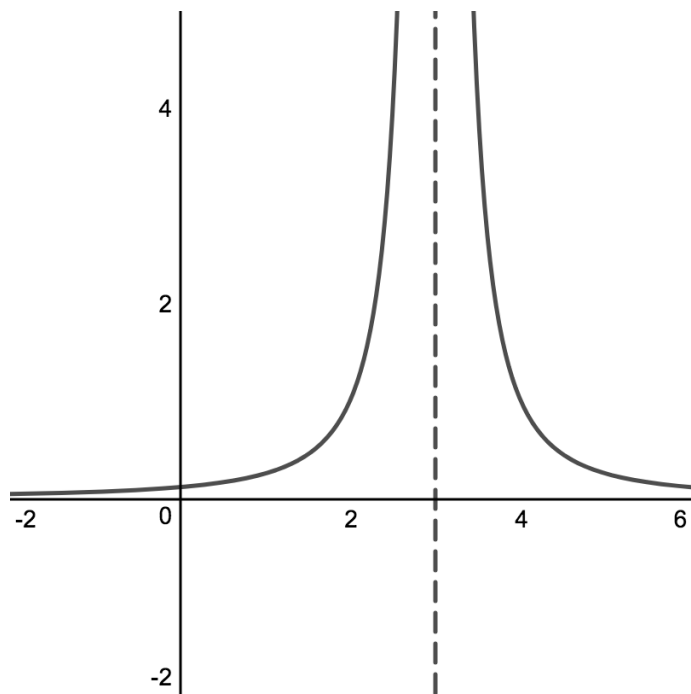
$$\lim_{x \rightarrow -1^+} f(x)$$

$$\lim_{x \rightarrow -1^-} f(x)$$

$$\lim_{x \rightarrow -1} f(x)$$

$$\lim_{x \rightarrow -\infty} f(x)$$

$$\lim_{x \rightarrow +\infty} f(x)$$



**Example 3:**

Find the following using the graph on the left.

$$f(3)$$

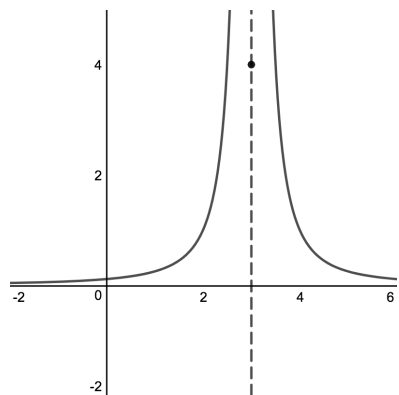
$$\lim_{x \rightarrow 3^+} f(x)$$

$$\lim_{x \rightarrow 3^-} f(x)$$

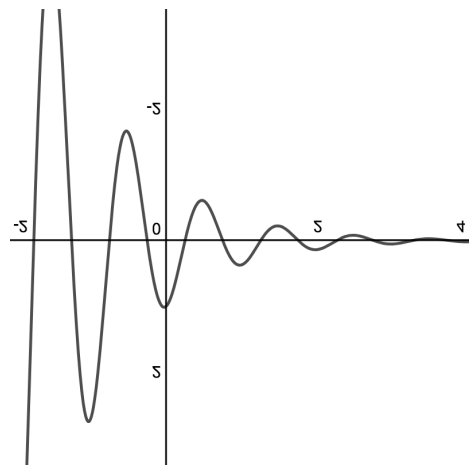
$$\lim_{x \rightarrow 3} f(x)$$

$$\lim_{x \rightarrow -\infty} f(x)$$

$$\lim_{x \rightarrow +\infty} f(x)$$

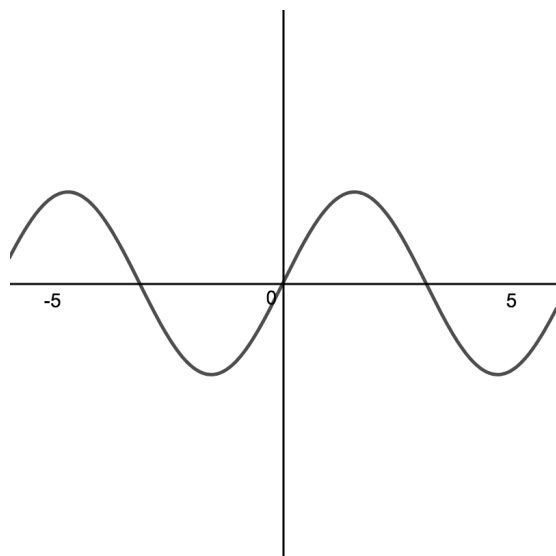


**Example 4:**  
Compare to Example 3. What changed?



**Example 5:**

$$\lim_{x \rightarrow +\infty} f(x)$$



**Example 6:**

$$\lim_{x \rightarrow +\infty} f(x)$$

$$\lim_{x \rightarrow -\infty} f(x)$$

## Finding Limits Numerically

Making a chart can be helpful when finding the limit.

The form  $0/0$  is called indeterminate.

**Example 7:** 
$$\lim_{x \rightarrow 2} \frac{x^2 - 3x + 2}{x - 2}$$

By direct substitution, you have the form  $0/0$  which is indeterminate. Make a chart.

x	1.9	1.99	1.99	2	2.001	2.01	2.1
f(x)	.9	.99	.999	??	1.001	1.01	1.1

**Example 8:** 
$$\lim_{x \rightarrow 0} \frac{x}{\sqrt{x+1}-1}$$

By direct substitution, you have the form  $0/0$  which is indeterminate. Make a chart.

x	-.01	-.001	-.0001	0	.0001	.001	.01
f(x)				??			

**Example 9:** 
$$\lim_{x \rightarrow 0} \frac{|x|}{x}$$

By direct substitution, you have the form  $0/0$  which is indeterminate. Make a chart.

x	-1	-.1	-.01	0	.01	.1	1
f(x)				??			

**Example 10:** 
$$\lim_{x \rightarrow 0} \frac{1}{x}$$

By direct substitution, you have the form  $1/0$  which is undefined? Make a chart.

x	-.01	-.001	-.0001	0	.0001	.001	.01
f(x)				??			

## Finding Limits Analytically

Try direct substitution

With indeterminate form, try to factor and cancel. If that doesn't work, try something else.

**Example 11:** Find the following limits.

a)  $\lim_{x \rightarrow -2} x^2$

b)  $\lim_{x \rightarrow 3} 5x - 4$

c)  $\lim_{x \rightarrow -5} 10$

d)  $\lim_{x \rightarrow 3} \sqrt[3]{2x^2 - 10}$

e)  $\lim_{x \rightarrow \pi} \sin x$

f)  $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x + 1}$

g)  $\lim_{x \rightarrow \pi} x \cos x$

**Example 12:** Find the following limits. Factor and cancel.

a)  $\lim_{x \rightarrow 1} \frac{x-1}{x^2-1}$

b)  $\lim_{x \rightarrow -1} \frac{x^3+1}{x^2-1}$

c)  $\lim_{x \rightarrow 4} \frac{x^2-2x-8}{x^2-5x+4}$

d)  $\lim_{x \rightarrow -2} \frac{x^2-3x-10}{x^2-4}$

e)  $\lim_{x \rightarrow 2} \frac{x^2-3x+2}{x-2}$

**Example 13:**

Find the following limits. Multiply by conjugate.

a) 
$$\lim_{x \rightarrow 0} \frac{x}{\sqrt{x+1}-1}$$

b) 
$$\lim_{x \rightarrow 0} \frac{\sqrt{x+16}-4}{x}$$

c) 
$$\lim_{x \rightarrow 16} \frac{\sqrt{x}-4}{x-16}$$

d) 
$$\lim_{x \rightarrow 3} \frac{\sqrt{x+1}-2}{x-3}$$

e) 
$$\lim_{x \rightarrow 3} \frac{1-\sqrt{x-2}}{x-3}$$

**Example 14:**

Find the following limits. Think about trig identities.

a) 
$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{1-\sin x}{\cos^2 x}$$

b) 
$$\lim_{x \rightarrow 0} \frac{\sin 2x}{\sin x}$$

### THEOREM 1—Limit Laws

If  $L$ ,  $M$ ,  $c$ , and  $k$  are real numbers and

$$\lim_{x \rightarrow c} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:*  $\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$
2. *Difference Rule:*  $\lim_{x \rightarrow c} (f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:*  $\lim_{x \rightarrow c} (k \cdot f(x)) = k \cdot L$
4. *Product Rule:*  $\lim_{x \rightarrow c} (f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:*  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:*  $\lim_{x \rightarrow c} [f(x)]^n = L^n, \quad n \text{ a positive integer}$
7. *Root Rule:*  $\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}, \quad n \text{ a positive integer}$

(If  $n$  is even, we assume that  $f(x) \geq 0$  for  $x$  in an interval containing  $c$ .)

### THEOREM 2—Limits of Polynomials

If  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ , then

$$\lim_{x \rightarrow c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

### THEOREM 3—Limits of Rational Functions

If  $P(x)$  and  $Q(x)$  are polynomials and  $Q(c) \neq 0$ , then

$$\lim_{x \rightarrow c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

**THEOREM 4—The Sandwich Theorem**

Suppose that  $g(x) \leq f(x) \leq h(x)$  for all  $x$  in some open interval containing  $c$ , except possibly at  $x = c$  itself. Suppose also that

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L.$$

Then  $\lim_{x \rightarrow c} f(x) = L$ .



## Section 2.2-2.4 Limit of a Function (Graphically, Numerically, Analytically)

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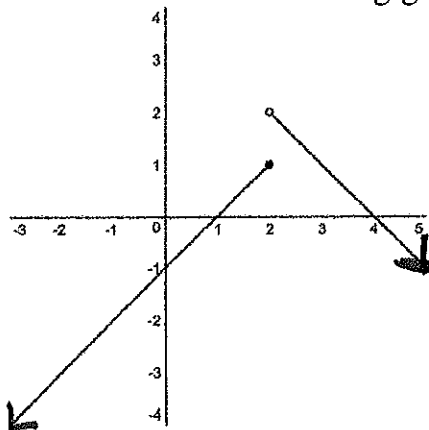
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$\lim_{x \rightarrow -\infty} f(x)$  and  $\lim_{x \rightarrow +\infty} f(x)$  refer to end behavior

Let's examine limits using graphs.



### **Example 1:**

Find the following using the graph on the left.

$$f(2) = 1$$

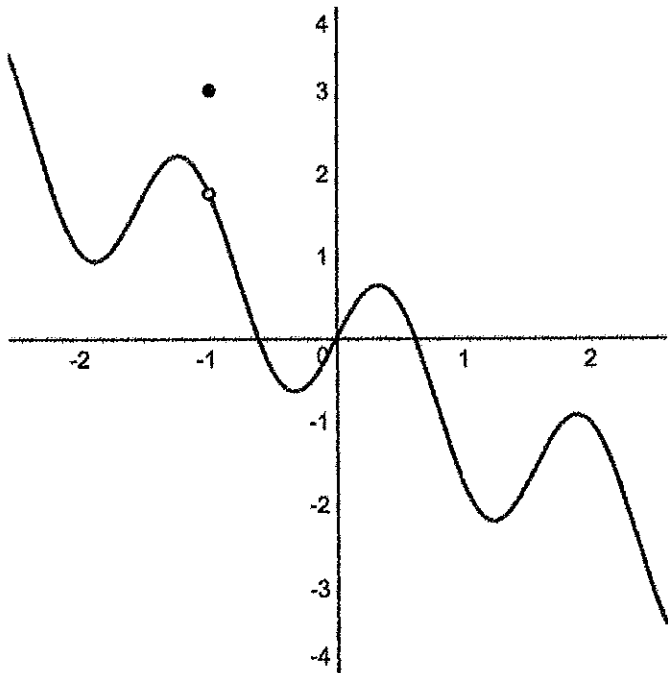
$$\lim_{x \rightarrow 2^+} f(x) = 2$$

$$\lim_{x \rightarrow 2^-} f(x) = 1$$

$$\lim_{x \rightarrow 2} f(x) \text{ DNE}$$

$$\lim_{x \rightarrow -\infty} f(x) = -\infty$$

$$\lim_{x \rightarrow +\infty} f(x) = -\infty$$



**Example 2:**

Find the following using the graph on the left.

$$f(-1) = 3$$

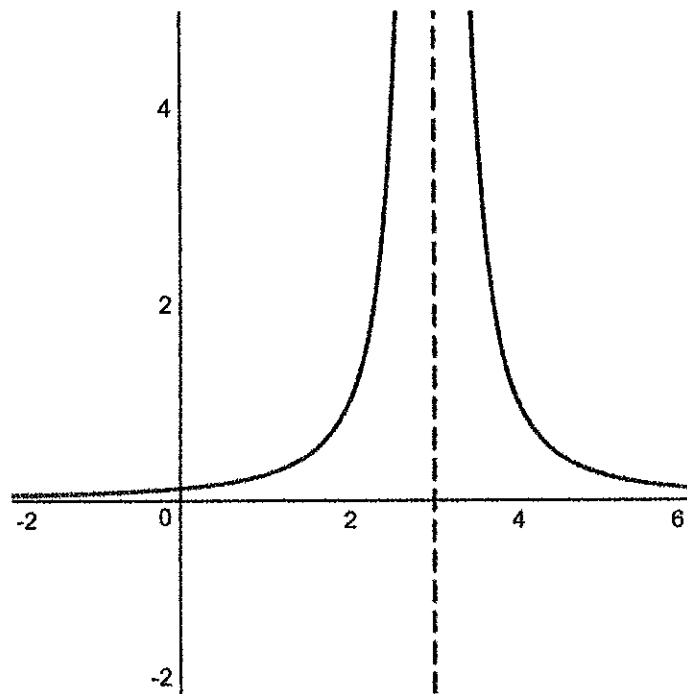
$$\lim_{x \rightarrow -1^+} f(x) = 2$$

$$\lim_{x \rightarrow -1^-} f(x) = 2$$

$$\lim_{x \rightarrow -1} f(x) = 2$$

$$\lim_{x \rightarrow -\infty} f(x) = +\infty$$

$$\lim_{x \rightarrow +\infty} f(x) = -\infty$$



**Example 3:**

Find the following using the graph on the left.

$$f(3) \text{ undefined}$$

vertical asymptote

$$\lim_{x \rightarrow 3^+} f(x) = +\infty$$

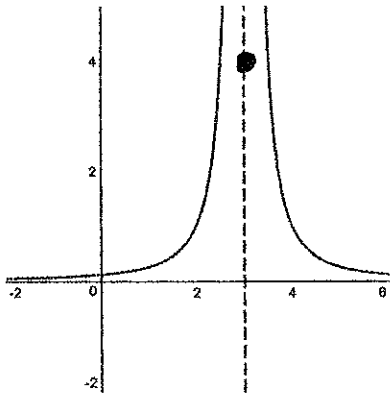
$$\lim_{x \rightarrow 3^-} f(x) = +\infty$$

$$\lim_{x \rightarrow 3} f(x) = +\infty$$

$$\lim_{x \rightarrow -\infty} f(x) = 0$$

$$\lim_{x \rightarrow +\infty} f(x) = 0$$

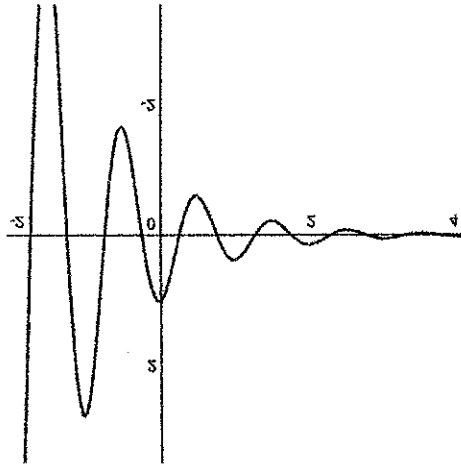
horizontal asymptote



**Example 4:**

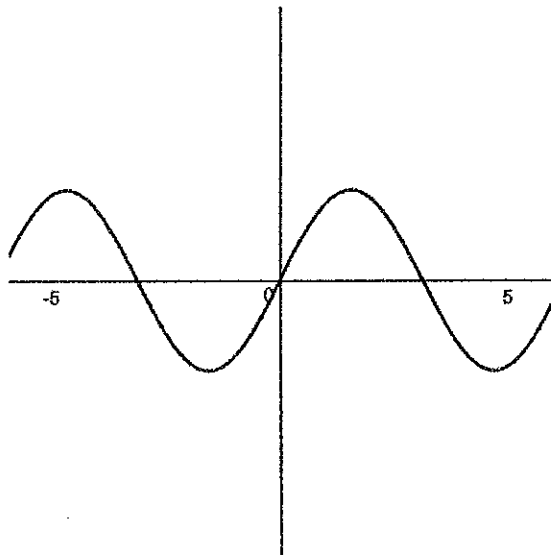
Compare to Example 3. What changed?

$$f(3) = 4$$



**Example 5:**

$$\lim_{x \rightarrow +\infty} f(x) = 0$$



**Example 6:**

$$\lim_{x \rightarrow +\infty} f(x) \text{ DNE}$$

$$\lim_{x \rightarrow -\infty} f(x) \text{ DNE}$$

because the graph oscillates between -1 and 1

## Finding Limits Numerically

Making a chart can be helpful when finding the limit.

The form 0/0 is called indeterminate.

**Example 7:**  $\lim_{x \rightarrow 2} \frac{x^2 - 3x + 2}{x - 2} = 1$

By direct substitution, you have the form 0/0 which is indeterminate. Make a chart.

x	1.9	1.99	1.99	2	2.001	2.01	2.1
f(x)	.9	.99	.999	??	1.001	1.01	1.1

$$\lim_{x \rightarrow 2^-} f(x) = 1$$

$$\lim_{x \rightarrow 2^+} f(x) = 1$$

**Example 8:**  $\lim_{x \rightarrow 0} \frac{x}{\sqrt{x+1}-1} = 2$

By direct substitution, you have the form 0/0 which is indeterminate. Make a chart.

x	-.01	-.001	-.0001	0	.0001	.001	.01
f(x)	1.995	1.9995	1.99995	??	2.0005	2.0005	2.005

$$\lim_{x \rightarrow 0^-} f(x) = 2$$

$$\lim_{x \rightarrow 0^+} f(x) = 2$$

**Example 9:**  $\lim_{x \rightarrow 0} \frac{|x|}{x} = \text{DNE}$

By direct substitution, you have the form 0/0 which is indeterminate. Make a chart.

x	-1	-.1	-.01	0	.01	.1	1
f(x)	-1	-1	-1	??	1	1	1

$$\lim_{x \rightarrow 0^-} f(x) = -1$$

$$\lim_{x \rightarrow 0^+} f(x) = 1$$

**Example 10:**  $\lim_{x \rightarrow 0} \frac{1}{x} = \text{DNE}$

By direct substitution, you have the form 1/0 which is undefined? Make a chart.

x	-.01	-.001	-.0001	0	.0001	.001	.01
f(x)	-100	-1000	-10000	??	10000	1000	100

$$\lim_{x \rightarrow 0^-} f(x) = -\infty$$

$$\lim_{x \rightarrow 0^+} f(x) = +\infty$$

## Finding Limits Analytically

Try direct substitution

With indeterminate form, try to factor and cancel. If that doesn't work, try something else.

**Example 11:** Find the following limits.

a)  $\lim_{x \rightarrow -2} x^2 = (-2)^2 = 4$

b)  $\lim_{x \rightarrow 3} 5x - 4 = 5(3) - 4 = 11$

c)  $\lim_{x \rightarrow -5} 10 = 10$

d)  $\lim_{x \rightarrow 3} \sqrt[3]{2x^2 - 10} = \sqrt[3]{2 \cdot 3^2 - 10} = \sqrt[3]{8} = 2$

e)  $\lim_{x \rightarrow \pi} \sin x = \sin \pi = 0$

f)  $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x + 1} = \frac{1 - 1}{1 + 1} = \frac{0}{2} = 0$

g)  $\lim_{x \rightarrow \pi} x \cos x = \pi \cos \pi = \pi(-1) = -\pi$

**Example 12:** Find the following limits. Factor and cancel.

$\frac{0}{0}$  a)  $\lim_{x \rightarrow 1} \frac{x-1}{x^2-1} = \lim_{x \rightarrow 1} \frac{x-1}{(x-1)(x+1)} = \lim_{x \rightarrow 1} \frac{1}{x+1} = \frac{1}{1+1} = \frac{1}{2}$

$\frac{0}{0}$  b)  $\lim_{x \rightarrow -1} \frac{x^3+1}{x^2-1} = \lim_{x \rightarrow -1} \frac{(x+1)(x^2-x+1)}{(x+1)(x-1)} = \lim_{x \rightarrow -1} \frac{x^2-x+1}{x-1} = \frac{-3}{2}$

$\frac{0}{0}$  c)  $\lim_{x \rightarrow 4} \frac{x^2-2x-8}{x^2-5x+4} = \lim_{x \rightarrow 4} \frac{(x-4)(x+2)}{(x-4)(x-1)} = \lim_{x \rightarrow 4} \frac{x+2}{x-1} = \frac{6}{3} = 2$

$\frac{0}{0}$  d)  $\lim_{x \rightarrow -2} \frac{x^2-3x-10}{x^2-4} = \lim_{x \rightarrow -2} \frac{(x-5)(x+2)}{(x+2)(x-2)} = \lim_{x \rightarrow -2} \frac{x-5}{x-2} = \frac{7}{4}$

$\frac{0}{0}$  e)  $\lim_{x \rightarrow 2} \frac{x^2-3x+2}{x-2} = \lim_{x \rightarrow 2} \frac{(x-2)(x-1)}{x-2} = \lim_{x \rightarrow 2} x-1 = 2-1 = 1$

**Example 13:**

Find the following limits. Multiply by conjugate.

$$\frac{0}{0} \text{ a) } \lim_{x \rightarrow 0} \frac{x}{\sqrt{x+1}-1} \cdot \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} = \lim_{x \rightarrow 0} \frac{x(\sqrt{x+1}+1)}{x+1+\sqrt{x+1}-\sqrt{x+1}-1} = \lim_{x \rightarrow 0} \frac{x(\sqrt{x+1}+1)}{x}$$

$$= \lim_{x \rightarrow 0} \sqrt{x+1}+1 = \sqrt{1}+1 = 2$$

$$\frac{0}{0} \text{ b) } \lim_{x \rightarrow 0} \frac{\sqrt{x+16}-4}{x} \cdot \frac{\sqrt{x+16}+4}{\sqrt{x+16}+4} = \lim_{x \rightarrow 0} \frac{x+16+4\sqrt{x+16}-4\sqrt{x+16}-16}{x(\sqrt{x+16}+4)}$$

$$= \lim_{x \rightarrow 0} \frac{x}{x(\sqrt{x+16}+4)} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+16}+4} = \frac{1}{\sqrt{16}+4} = \frac{1}{8}$$

$$\frac{0}{0} \text{ c) } \lim_{x \rightarrow 16} \frac{\sqrt{x}-4}{x-16} \cdot \frac{\sqrt{x}+4}{\sqrt{x}+4} = \lim_{x \rightarrow 16} \frac{x-16}{(x-16)(\sqrt{x}+4)} = \lim_{x \rightarrow 16} \frac{1}{\sqrt{x}+4} = \frac{1}{\sqrt{16}+4} = \frac{1}{8}$$

$$\frac{0}{0} \text{ d) } \lim_{x \rightarrow 3} \frac{\sqrt{x+1}-2}{x-3} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} = \lim_{x \rightarrow 3} \frac{x+1-4}{(x-3)(\sqrt{x+1}+2)} = \lim_{x \rightarrow 3} \frac{1}{\sqrt{x+1}+2} = \frac{1}{4}$$

$$\frac{0}{0} \text{ e) } \lim_{x \rightarrow 3} \frac{1-\sqrt{x-2}}{x-3} \cdot \frac{1+\sqrt{x-2}}{1+\sqrt{x-2}} = \lim_{x \rightarrow 3} \frac{1-(x-2)}{(x-3)(1+\sqrt{x-2})} = \lim_{x \rightarrow 3} \frac{3-x}{(x-3)(1+\sqrt{x-2})}$$

$$= \lim_{x \rightarrow 3} \frac{-1}{1+\sqrt{x-2}} = \frac{-1}{1+\sqrt{1}} = -\frac{1}{2}$$

**Example 14:**

Find the following limits. Think about trig identities.

$$\frac{0}{0} \text{ a) } \lim_{x \rightarrow \frac{\pi}{2}} \frac{1-\sin x}{\cos^2 x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{1-\sin x}{1-\sin^2 x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{1-\sin x}{(1-\sin x)(1+\sin x)}$$

$$= \lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{1+\sin x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{1+\sin \frac{\pi}{2}} = \frac{1}{1+1} = \frac{1}{2}$$

$$\frac{0}{0} \text{ b) } \lim_{x \rightarrow 0} \frac{\sin 2x}{\sin x} = \lim_{x \rightarrow 0} \frac{2 \sin x \cos x}{\sin x} = \lim_{x \rightarrow 0} 2 \cos x = 2 \cos 0 = 2(1) = 2$$