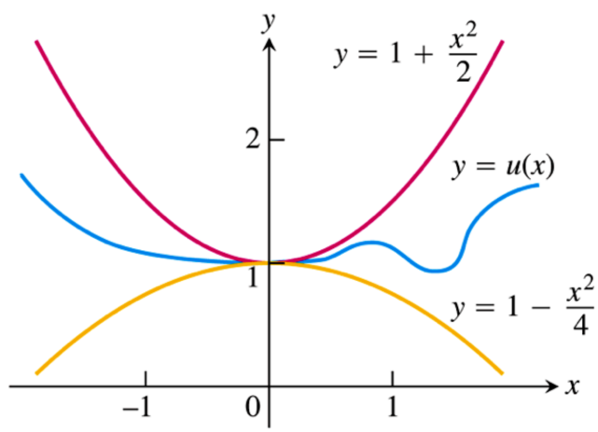
Section 2.2-2.4 Limit of a Function
(Sandwich, Squeeze, or Pinch Theorem)

THEOREM 4—The Sandwich Theorem Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

Example 1: Find $\lim_{x \to 0} u(x)$



The graph of $u(x)$ is sandwiched between $y = 1 + \frac{x^2}{2}$ and $y = 1 - \frac{x^2}{4}$.

$$\lim_{x \to 0} 1 + \frac{x^2}{2} = 1$$

$$\lim_{x \to 0} 1 - \frac{x^2}{4} = 1$$

So $\lim_{x \to 0} u(x) = 1$

The Sandwich Theorem (also known as the Squeeze or Pinching Theorem) can be used to prove the following.

THEOREM 7—Limit of the Ratio $\sin\theta/\theta$ as $\theta \to 0$

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \qquad (1)$$

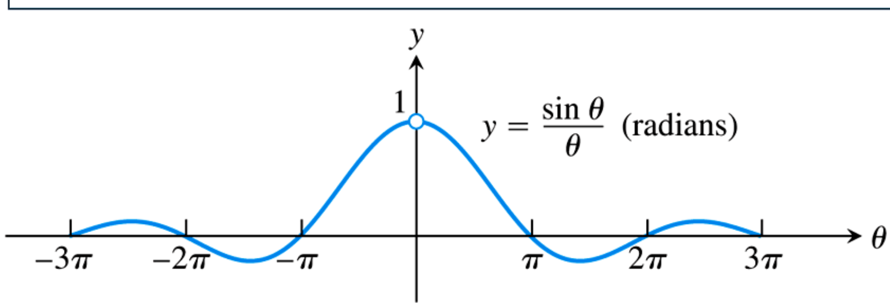


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FIGURE 2.32 The graph of $f(\theta) = (\sin\theta)/\theta$ suggests that the right- and left-hand limits as θ approaches 0 are both 1.

Example 2: Find the following limits using Theorem 7.

a) $\lim_{x\to 0} \frac{\sin 3x}{3x}$

b) $\lim_{x\to 0} \frac{5x}{\sin 5x}$

c) $\lim_{x\to 0} \frac{\sin 4x}{x}$

d) $\lim_{x\to 0} \frac{\sin 18x}{17x}$

e) $\lim_{x\to 0} \frac{\sin 3x}{\sin 4x}$

f) $\lim_{x\to 0} \frac{\tan 2x}{x}$

g) $\lim_{x\to 0} \frac{\sin 2x \cot 3x}{\mathrm{x}\cot 4x}$

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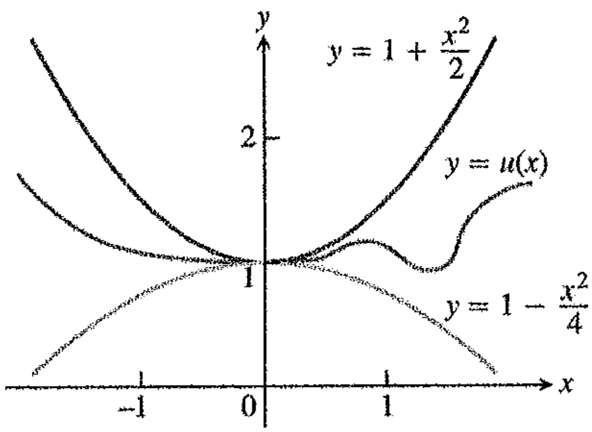

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$$\lim_{x \to 0} 1 + \frac{x^2}{2} = 1$$

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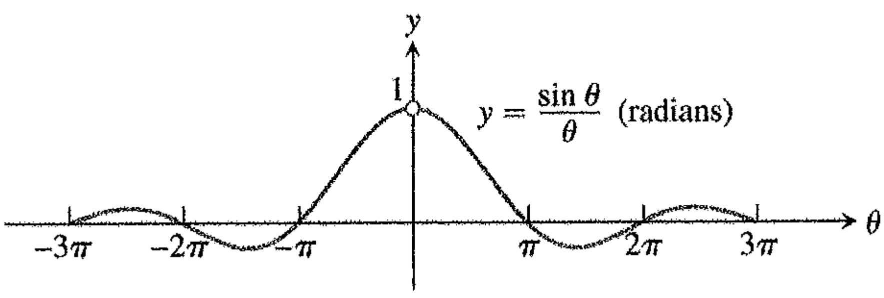

FIGURE 2.32 The graph of $f(\theta) = (\sin\theta)/\theta$ suggests that the right- and left-hand limits as θ approaches 0 are both 1.

Example 2: Find the following limits using Theorem 7.

a) $\lim_{x \to 0} \frac{\sin 3x}{3x} = 1$

b) $\lim_{x \to 0} \frac{5x}{\sin 5x} = 1$

c) $\lim_{x \to 0} \frac{\sin 4x}{x} = \lim_{x \to 0} \frac{4 \sin 4x}{4x} = 4 \cdot 1 = 4$

d) $\lim_{x \to 0} \frac{\sin 18x}{17x} = \lim_{x \to 0} \frac{18 \sin 18x}{17 \cdot 18x} = \frac{18}{17} \cdot 1 = \frac{18}{17}$

e) $\lim_{x \to 0} \frac{\sin 3x}{\sin 4x} = \lim_{x \to 0} \frac{3x \sin 3x}{3x} \cdot \frac{4x}{4x \sin 4x} = \lim_{x \to 0} \frac{3x}{4x} \cdot \frac{\sin 3x}{3x} \cdot \frac{4x}{\sin 4x}$

$= \frac{3}{4} \cdot 1 \cdot 1 = 3/4$

f) $\lim_{x \to 0} \frac{\tan 2x}{x} = \lim_{x \to 0} \frac{\sin 2x}{x \cos 2x} = \lim_{x \to 0} \frac{2 \sin 2x}{2x \cos 2x} = \lim_{x \to 0} \frac{2}{\cos 2x} \cdot \frac{\sin 2x}{2x}$

$= \frac{2}{1} \cdot 1 = 2$

g) $\lim_{x \to 0} \frac{\sin 2x \cot 3x}{x \cot 4x}$

$= \lim_{x \to 0} \frac{\sin 2x}{x} \frac{\cos 3x}{\sin 3x} \frac{\sin 4x}{\cos 4x}$

$= \lim_{x \to 0} \frac{2 \sin 2x \cdot 3x \cdot \cos 3x \cdot 4x \sin 4x}{2x \cdot 3x \sin 3x \cdot \cos 4x \cdot 4x}$

$= \lim_{x \to 0} 2 \cdot \frac{4x}{3x} \cdot \frac{\cos 3x}{\cos 4x} \cdot \frac{\sin 2x}{2x} \cdot \frac{3x}{\sin 3x} \cdot \frac{\sin 4x}{4x}$

$= 2 \cdot \frac{4}{3} \cdot 1 \cdot 1 \cdot 1 \cdot 1 = \frac{8}{3}$