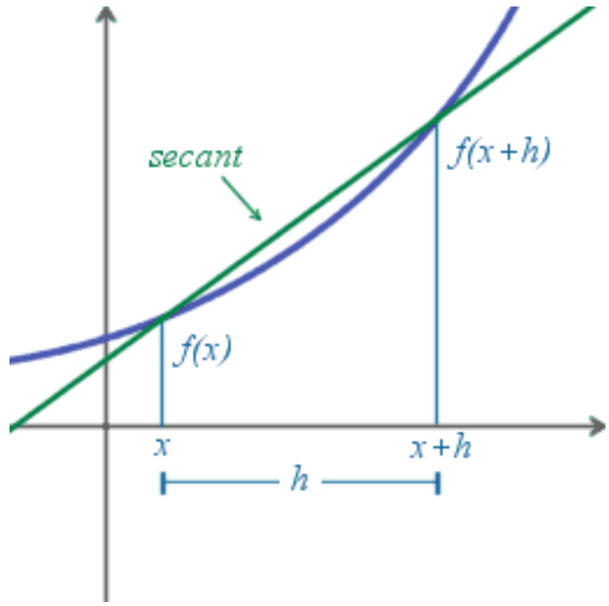
Section 3.1-3.2 Tangent Lines and the Derivative

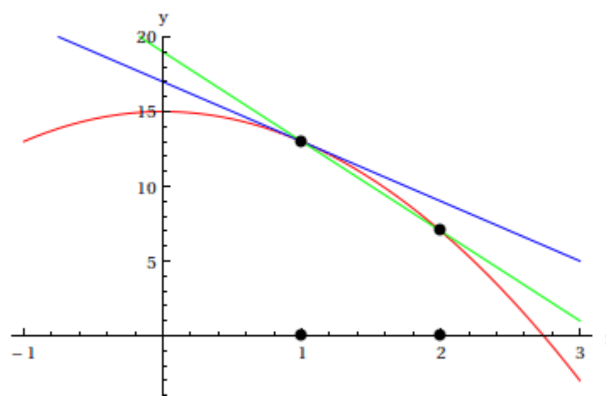
What's a secant line to a curve?
A straight line between two points on the curve.
The diagram shows two points on the curve
(x, f(x)) and (x+h, f(x+h))



What's the slope of the secant line?

$$m = \frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{x+h-x} = \frac{f(x+h) - f(x)}{x}$$

What's the tangent line?
A straight line that touches the curve at one point.
The green line is a secant line. The blue line is a tangent line to the curve at x=1.



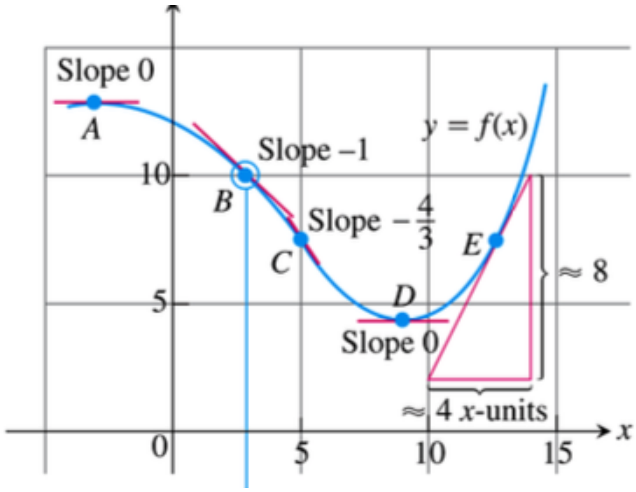
What's the slope of the tangent line?
Imagine that the point (x+h, f(x+h)) slides to the point (x, f(x)) so that the distance h goes to 0.

$$m_{tan} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{x}$$

The following are all interpretations for the limit of the difference quotient

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$
2. The slope of the tangent line to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at the $x = x_0$
4. The derivative $f'(x_0)$ at $x = x_0$



Example 1:
Use a grade and a straight edge to make a rough estimate of the slope of the curve at the points A, B, C, D, and E.

Example 2: Find the slope of the curve $f(x) = 3x + 4$. Explain your result with the graph f(x)

DEFINITION The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

Example 3: Given $\mathrm{f}(x) = x^2 + 1$.

a) Find $f'(x)$

b) Find the slope of the tangent line to $f(x)$ when x=-2, x=0, and x=2.

c) Compare slopes found in b to the graph of $f(x)$.

d) Graph $f'(x)$ and draw conclusions.

Example 4: Given $f(x) = x^2 - 8x + 9$

a) Find $f'(x)$

b) Find $f'(3)$

c) Find $f(3)$

d) Find the equation of the tangent line to the graph of f(x) at x=3.

e) Graph f(x) and the line from part d.

Example 5: Given $f(x) = \frac{1}{x}$

a) Find $f'(x)$

b) Find $f'(2)$

c) Find the equation of the tangent line to the graph of f(x) at x=2.

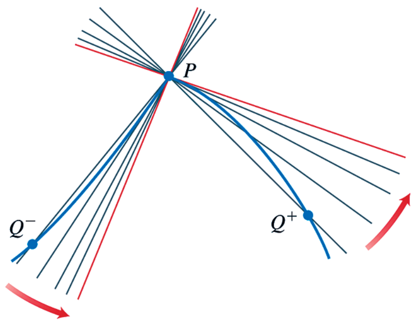
Example 6: Given $f(x) = \sqrt{x}$

a) Find $f'(x)$

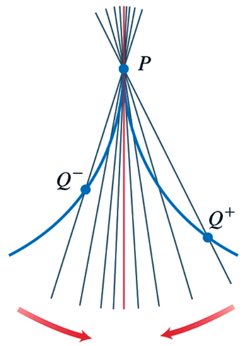
b) Find the slope of the tangent line at x=9

c) Find the equation of the tangent line to the graph of f(x) at x=9.

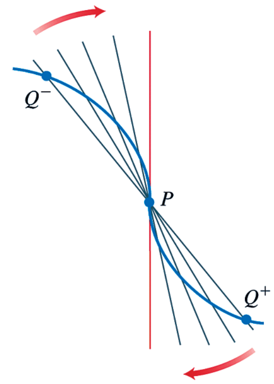
When Does a Function NOT Have a Derivative at a Point?



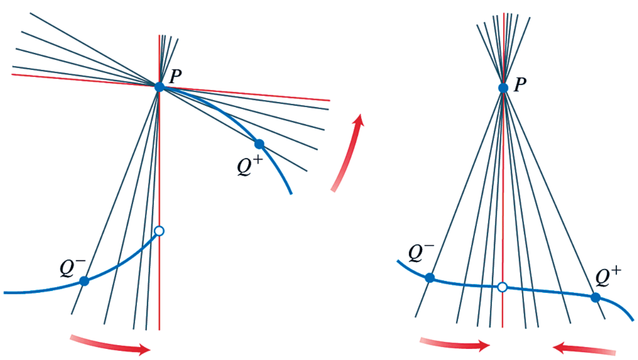
1. a *corner*, where the one-sided derivatives differ



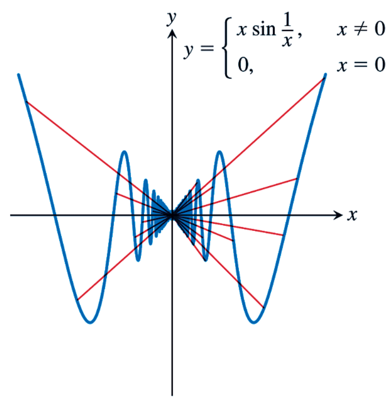
2. a *cusp*, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other



3. a *vertical tangent line*, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$)



4. a *discontinuity* (two examples shown)



5. wild oscillation

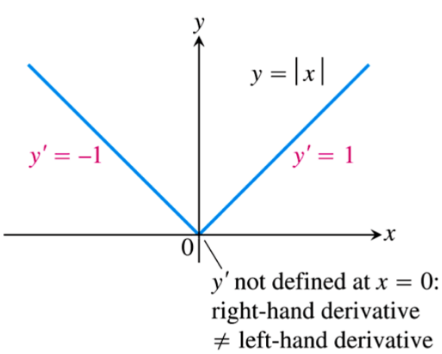


FIGURE 3.8 The function $y = |x|$ is not differentiable at the origin where the graph has a "corner"

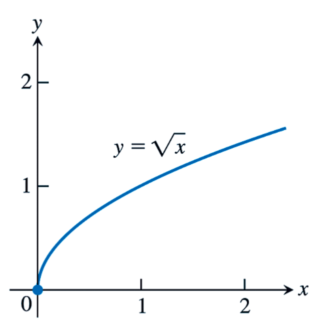
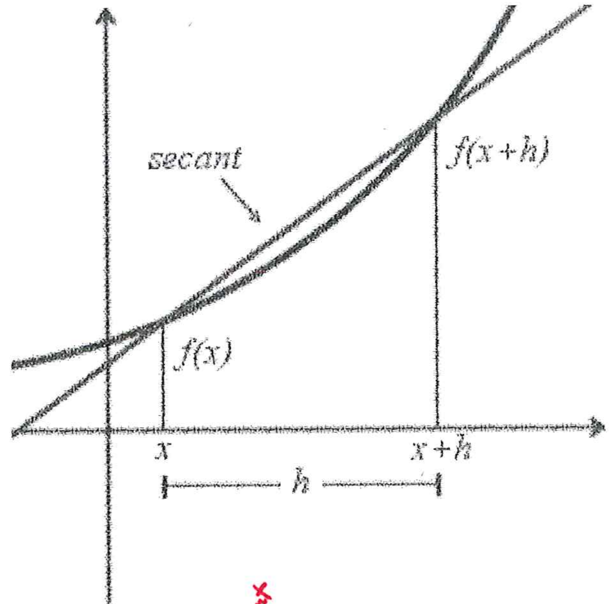


FIGURE 3.9 The square root function is not differentiable at $x = 0$, where the graph of the function has a vertical tangent line.

> **THEOREM 1—Differentiability Implies Continuity** If f has a derivative at $x = c$, then f is continuous at $x = c$.

Section 3.1-3.2 Tangent Lines and the Derivative

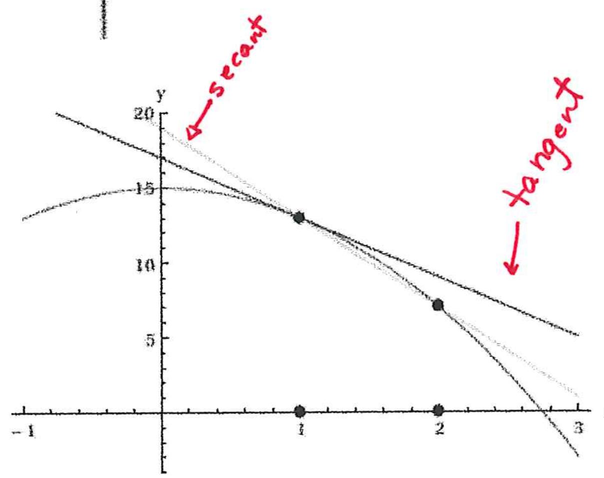
What's a secant line to a curve?
A straight line between two points on the curve.
The diagram shows two points on the curve
(x, f(x)) and (x+h, f(x+h))



What's the slope of the secant line?

$$m = \frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{x + h - x} = \frac{f(x+h) - f(x)}{x}$$

What's the tangent line?
A straight line that touches the curve at one point.
The green line is a secant line. The blue line is a tangent line to the curve at x=1.



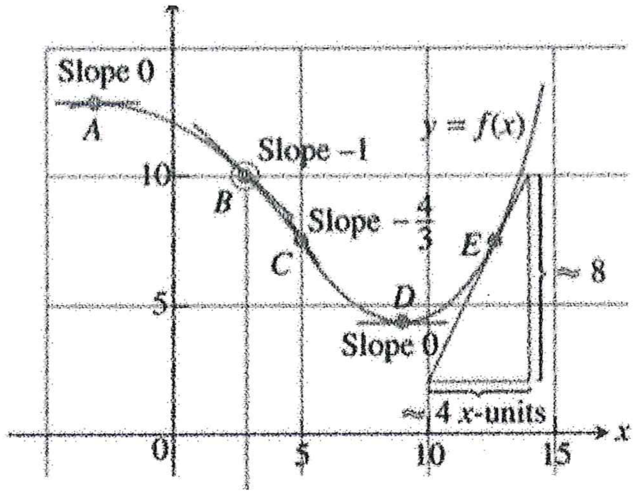
What's the slope of the tangent line?
Imagine that the point (x+h, f(x+h)) slides to the point (x, f(x)) so that the distance h goes to 0.

$$m_{tan} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{x}$$

The following are all interpretations for the limit of the difference quotient

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$
2. The slope of the tangent line to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at the $x = x_0$
4. The derivative $f'(x_0)$ at $x = x_0$



Example 1:
Use a grade and a straight edge to make a rough estimate of the slope of the curve at the points A, B, C, D, and E.

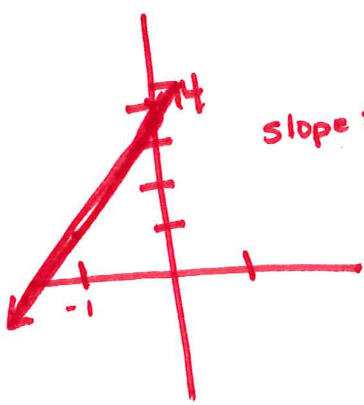
Example 2: Find the slope of the curve $f(x) = 3x + 4$. Explain your result with the graph f(x)

$$m = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{3(x+h)+4-(3x+4)}{h} = \lim_{h\to 0} \frac{3x+3h+4-3x-4}{h}$$

$$= \lim_{h\to 0} \frac{3h}{h} = \lim_{h\to 0} 3 = 3$$

Slope of curve = 3

$f(x) = 3x+4$ is a line with slope of 3 and intercept of 4.



slope = 3

DEFINITION The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

Example 3: Given $f(x) = x^2 + 1$.

a) Find $f'(x)$

b) Find the slope of the tangent line to $f(x)$ when x=-2, x=0, and x=2.

c) Compare slopes found in b to the graph of $f(x)$.

d) Graph $f'(x)$ and draw conclusions.

a) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 + 1 - (x^2+1)}{h}$

$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 1 - x^2 - 1}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h}$

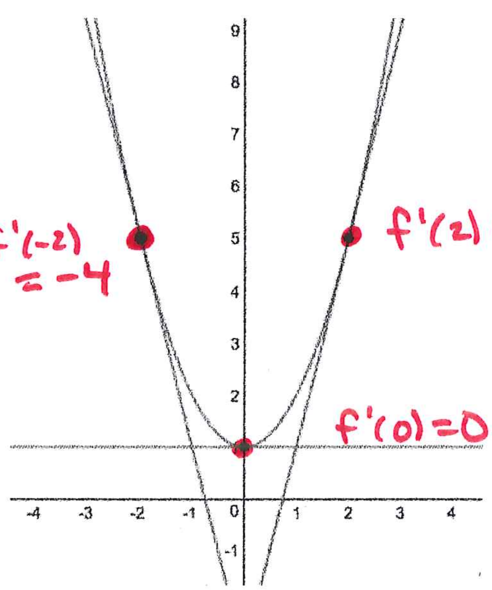
$= \lim_{h \to 0} 2x + h = 2x + 0 = 2x$

b) $f'(-2) = 2(-2) = -4$

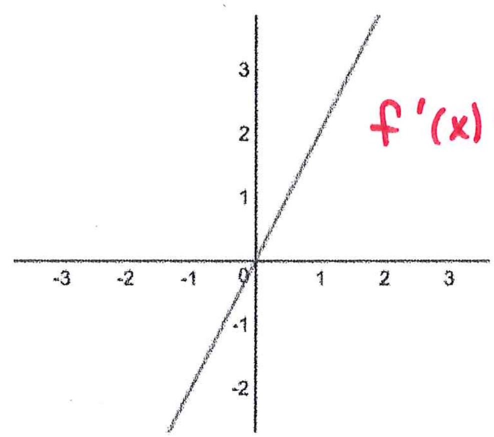
$f'(0) = 2(0) = 0$

$f'(2) = 2(2) = 4$

c)



d)



Example 4: Given $f(x) = x^2 - 8x + 9$

a) Find $f'(x)$

b) Find $f'(3)$

c) Find $f(3)$

d) Find the equation of the tangent line to the graph of f(x) at x=3.

e) Graph f(x) and the line from part d.

a) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - 8(x+h) + 9 - (x^2 - 8x + 9)}{h}$

$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 8x - 8h + 9 - x^2 + 8x - 9}{h}$

$= \lim_{h \to 0} \frac{2xh + h^2 - 8h}{h} = \lim_{h \to 0} 2x + h - 8 = 2x - 8$

b) $f'(3) = 2(3) - 8 = -2$ ← slope

c) $f(3) = 3^2 - 8(3) + 9 = -6$ ← y value when $x = 3$

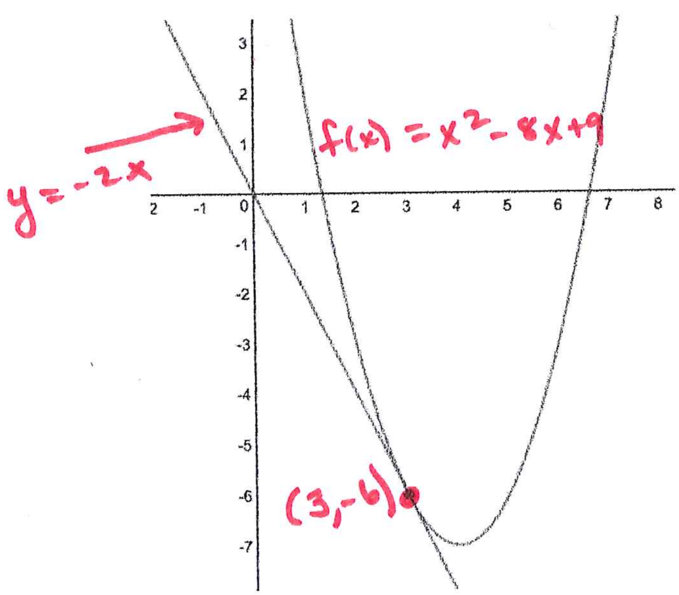
d) $y - y_1 = m(x - x_1)$

$y - (-6) = -2(x - 3)$

$y + 6 = -2x + 6$

$y = -2x$

e)



Example 5: Given $f(x) = \frac{1}{x}$

a) Find $f'(x)$

b) Find $f'(2)$

c) Find the equation of the tangent line to the graph of f(x) at x=2.

a) $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \cdot \frac{x(x+h)}{x(x+h)}$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{xh(x+h)} = \lim_{h \to 0} \frac{-h}{xh(x+h)}$$

$$= \lim_{h \to 0} \frac{-1}{x(x+h)} = \frac{-1}{x(x+0)} = \frac{-1}{x^2}$$

b) $$f'(2) = \frac{-1}{2^2} = -1/4$$

c) $f(2) = 1/2$

$$y - y_1 = m(x - x_1)$$

$$y - 1/2 = -1/4(x - 2)$$

$$y - 1/2 = -1/4x + \frac{1}{2}$$

$$y = -\frac{1}{4}x + 1$$

Example 6: Given $f(x) = \sqrt{x}$

a) Find $f'(x)$

b) Find the slope of the tangent line at x=9

c) Find the equation of the tangent line to the graph of f(x) at x=9.

a) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$

$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$

$= \lim_{h \to 0} \frac{(x+h) - (x)}{h(\sqrt{x+h} + \sqrt{x})}$

$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$

$= \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$

b) $f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{2(3)} = \frac{1}{6}$

c) $f(9) = \sqrt{9} = 3$

$y - y_1 = m(x - x_1)$

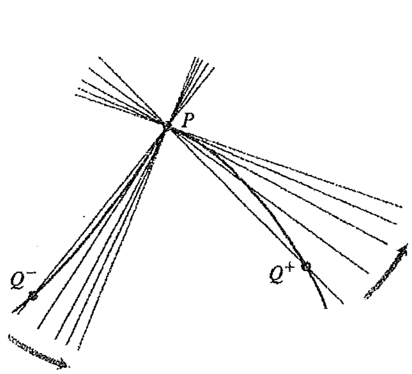
$y - 3 = 1/6(x - 9)$

$y - 3 = \frac{1}{6}x - \frac{9}{6}$

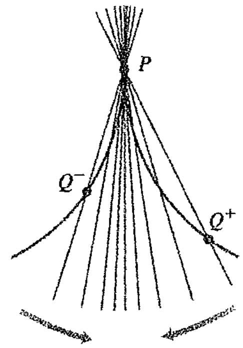
$y - 3 = \frac{1}{6}x - \frac{3}{2}$

$y = \frac{1}{6}x + \frac{3}{2}$

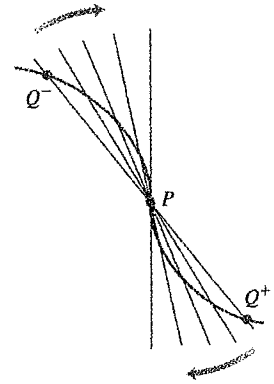
When Does a Function NOT Have a Derivative at a Point?



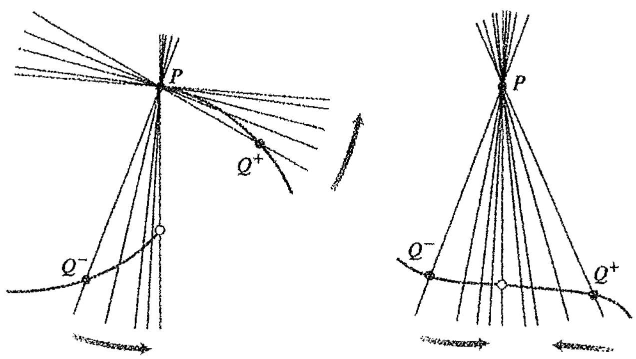
1. a *corner*, where the one-sided derivatives differ



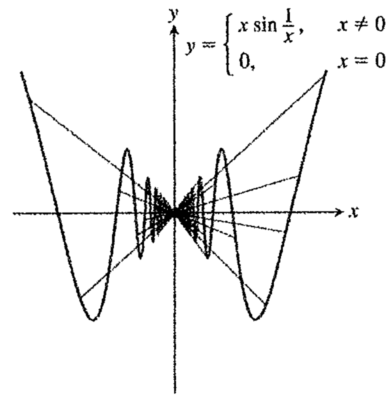
2. a *cusp*, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other



3. a *vertical tangent line*, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$)



4. a *discontinuity* (two examples shown)



5. wild oscillation

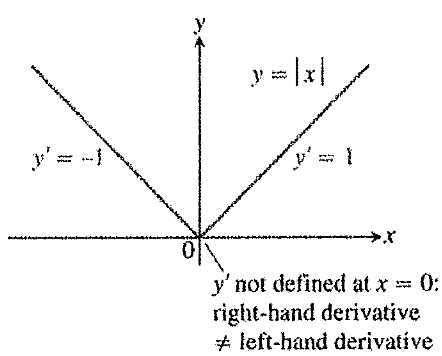


FIGURE 3.8 The function $y = |x|$ is not differentiable at the origin where the graph has a "corner"

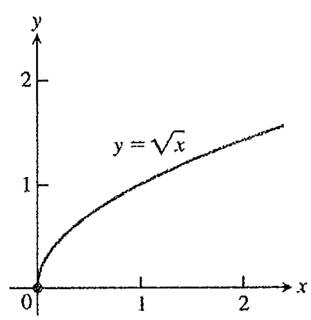


FIGURE 3.9 The square root function is not differentiable at $x = 0$, where the graph of the function has a vertical tangent line.

THEOREM 1—Differentiability Implies Continuity If f has a derivative at $x = c$, then f is continuous at $x = c$.