Section 3.3 Differentiation Rules

Constant Rule:
The derivative of every constant function is zero.
If $f(x) = c$, then $f'(x) = 0$.

Derivative of a Constant Function
If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Example 1: Find $f'(x)$ when $f(x) = 2$ and interpret your result.

Linear Functions:
If $f(x) = mx + b$, then If $f'(x) = m$.

Example 2: Find $f'(x)$ when $f(x) = 3x + 1$ and interpret your result.

Power Rule:
Subtract 1 from the exponent and multiply the result by the original exponent.
If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

Power Rule (General Version)
If n is any real number, then

$$\frac{d}{dx}x^n = nx^{n-1},$$

for all x where the powers x^n and x^{n-1} are defined.

Example 3: Find $f'(x)$ for the following functions using the power rule.
a) $f(x) = x^2$

b) $f(x) = x^3$

c) $f(x) = x^{100}$

d) $f(x) = \frac{1}{x}$

e) $f(x) = \frac{1}{x^2}$

f) $f(x) = \sqrt{x}$

g) $f(x) = \sqrt[3]{x}$

h) $f(x) = \sqrt[5]{x^4}$

Constant Multiple Rule:
When a differentiable function is multiplied by a constant, its derivative is multiplied by the same constant.
If $f(x) = c \bullet x^n$, then $f'(x) = c \bullet nx^{n-1}$.

Derivative Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

Example 4: Find $f'(x)$ for the following functions.
a) $f(x) = 5x^2$

b) $f(x) = -6x^4$

c) $f(x) = \frac{5}{x^2}$

Sum or Difference of Functions:
The derivative of the sum of two differentiable functions is the sum of their derivatives. The derivative of the difference of two differentiable functions is the difference of their derivatives.

Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Example 5: Find $f'(x)$ for the following functions.
a) $f(x) = x^4 + x^3 - x^2 + 10$

b) $f(x) = 4\sqrt[2]{x} - 5x^{-4} + 6$

c) $f(x) = x^5 - x^7 - \sqrt[3]{x} + \frac{1}{x^3}$

d) $f(x) = (x + 4)(x^2 - 1)$

e) $f(x) = \frac{3x^2+x-2}{x}$

Product Rule:
If $f(x) = u \bullet v$, then $f'(x) = u \bullet v' + v \bullet u'$.
If $f(x) = first \bullet second$,
then $f'(x) = first \bullet derivative(second) + second \bullet derivative(first)$.

Derivative Product Rule
If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + \frac{du}{dx}v.$$

Example 6: Find $f'(x)$ for the following functions. Don't simplify.
a) $f(x) = (x^4 - 2x^3 - 7)(3x^2 - 5x)$

b) $f(x) = (x^2 + 4x - 11)\left(7x^3 - \sqrt{x}\right)$

Quotient Rule:

If $f(x) = \frac{u}{v}$, then $f'(x) = \frac{v \bullet u' - u \bullet v'}{v^2}$

If $f(x) = \frac{hi}{lo}$, then $f'(x) = \frac{lo \bullet hi' - hi \bullet lo'}{lo^2}$ or "lo d. hi – hi d.l o over the square of what's below"

Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Example 7: Find $f'(x)$ for the following functions. Don't simplify.

a) $f(x) = \frac{1+x^2}{x^3+1}$

b) $f(x) = (x+5)(2x-7)^{-1}$

Applications of Derivatives

Example 8: Find the equation of the tangent line to the graph of $f(x) = x^3 - 2x + 1$ at x=2.

Example 9: Find the points (x,y) on the graph of $f(x) = -x^3 + x^2 + 5x - 1$ at which the tangent line is horizontal.

How to Read the Symbols for Derivatives

y'	"*y* prime"
y''	"*y* double prime"
$\frac{d^2y}{dx^2}$	"*d* squared *y* *dx* squared"
y'''	"*y* triple prime"
$y^{(n)}$	"*y* super *n*"
$\frac{d^ny}{dx^n}$	"*d* to the *n* of *y* by *dx* to the *n*"
D^n	"*d* to the *n*"

Example 10: Find first four derivatives of $f(x) = x^3 - 3x^2 + 2$.

Section 3.3 Differentiation Rules

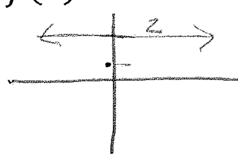
Constant Rule:
The derivative of every constant function is zero.
If $f(x) = c$, then $f'(x) = 0$.

> **Derivative of a Constant Function**
> If f has the constant value $f(x) = c$, then
> $$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Example 1: Find $f'(x)$ when $f(x) = 2$ and interpret your result.

$f(x) = 2$

$f'(x) = 0$



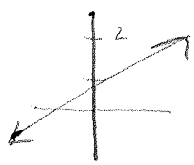
$f(x) = 2$ is horizontal line
slope is zero

Linear Functions:
If $f(x) = mx + b$, then If $f'(x) = m$.

Example 2: Find $f'(x)$ when $f(x) = 3x + 1$ and interpret your result.

$f(x) = 3x + 1$

$f'(x) = 3$



$f(x) = 3x+1$ is line with
slope of 3

Power Rule:
Subtract 1 from the exponent and multiply the result by the original exponent.
If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

> **Power Rule (General Version)**
> If n is any real number, then
> $$\frac{d}{dx}x^n = nx^{n-1},$$
> for all x where the powers x^n and x^{n-1} are defined.

Example 3: Find $f'(x)$ for the following functions using the power rule.

a) $f(x) = x^2$ $\quad f'(x) = 2x$

b) $f(x) = x^3$ $\quad f'(x) = 3x^2$

c) $f(x) = x^{100}$ $\quad f'(x) = 100x^{99}$

d) $f(x) = \frac{1}{x} = x^{-1}$ $\quad f'(x) = -x^{-2} = \frac{-1}{x^2}$

e) $f(x) = \frac{1}{x^2} = X^{-2}$ $\qquad f'(x) = -2X^{-3} = \frac{-2}{X^3}$

f) $f(x) = \sqrt{x} = X^{1/2}$ $\qquad f'(x) = \frac{1}{2}X^{-1/2} = \frac{1}{2\sqrt{X}}$

g) $f(x) = \sqrt[3]{x} = X^{1/3}$ $\qquad f'(x) = \frac{1}{3}X^{-2/3} = \frac{1}{3\sqrt[3]{X^2}}$

h) $f(x) = \sqrt[5]{x^4} = X^{\frac{4}{5}}$ $\qquad f'(x) = \frac{4}{5}X^{-\frac{1}{5}} = \frac{4}{5\sqrt[5]{X}}$

Constant Multiple Rule:

When a differentiable function is multiplied by a constant, its derivative is multiplied by the same constant.

If $f(x) = c \bullet x^n$, then $f'(x) = c \bullet nx^{n-1}$.

> **Derivative Constant Multiple Rule**
>
> If u is a differentiable function of x, and c is a constant, then
>
> $$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

Example 4: Find $f'(x)$ for the following functions.

a) $f(x) = 5x^2$ $\qquad f'(x) = 5 \cdot 2X = 10X$

b) $f(x) = -6x^4$ $\qquad f'(x) = -6 \cdot 4X^3 = -24X^3$

c) $f(x) = \frac{5}{x^2} = 5X^{-2}$ $\qquad f'(x) = 5 \cdot -2X^{-3} = -10X^{-3} = \frac{-10}{X^3}$

Sum or Difference of Functions:

The derivative of the sum of two differentiable functions is the sum of their derivatives. The derivative of the difference of two differentiable functions is the difference of their derivatives.

> **Derivative Sum Rule**
>
> If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,
>
> $$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Example 5: Find $f'(x)$ for the following functions.

a) $f(x) = x^4 + x^3 - x^2 + 10$

$$f'(x) = 4x^3 + 3x^2 - 2x$$

b) $f(x) = 4\sqrt[2]{x} - 5x^{-4} + 6$

$= 4x^{1/2} - 5x^{-4} + 6$

$$f'(x) = 2x^{-1/2} + 20x^{-5}$$

c) $f(x) = x^5 - x^7 - \sqrt[3]{x} + \frac{1}{x^3}$

$$f'(x) = 5x^4 - 7x^6 - \frac{1}{3}x^{-2/3} - 3x^{-4}$$

d) $f(x) = (x+4)(x^2 - 1)$

$= x^3 + 4x^2 - x - 4$

$$f'(x) = 3x^2 + 8x - 1$$

e) $f(x) = \frac{3x^2 + x - 2}{x}$

$= \frac{3x^2}{x} + \frac{x}{x} - \frac{2}{x}$

$= 3x + 1 - 2x^{-1}$

$$f'(x) = 3 + 2x^{-2}$$

Product Rule:

If $f(x) = u \bullet v$, then $f'(x) = u \bullet v' + v \bullet u'$.

If $f(x) = first \bullet second$,

then $f'(x) = first \bullet derivative(second) + second \bullet derivative(first)$.

> **Derivative Product Rule**
>
> If u and v are differentiable at x, then so is their product uv, and
>
> $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + \frac{du}{dx}v.$$

Example 6: Find $f'(x)$ for the following functions. Don't simplify.

a) $f(x) = (x^4 - 2x^3 - 7)(3x^2 - 5x)$

$$f'(x) = (x^4 - 2x^3 - 7)(3x^2 - 5x)' + (3x^2 - 5x)(x^4 - 2x^3 - 7)'$$

$$= (x^4 - 2x^3 - 7)(6x - 5) + (3x^2 - 5x)(4x^3 - 6x^2)$$

b) $f(x) = (x^2 + 4x - 11)(7x^3 - \sqrt{x})$

$$f'(x) = (x^2 + 4x - 11)(7x^3 - x^{1/2})' + (7x^3 - x^{1/2})(x^2 + 4x - 11)'$$

$$= (x^2 + 4x - 11)\left(21x^2 - \frac{1}{2}x^{-1/2}\right) + (7x^3 - x^{1/2})(2x + 4)$$

Quotient Rule:

If $f(x) = \frac{u}{v}$, then $f'(x) = \frac{v \bullet u' - u \bullet v\prime}{v^2}$

If $f(x) = \frac{hi}{lo}$, then $f'(x) = \frac{lo \bullet hi' - hi \bullet lo\prime}{lo^2}$ or "lo d. hi – hi d.l o over the square of what's below"

Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Example 7: Find $f'(x)$ for the following functions. Don't simplify.

a) $f(x) = \frac{1+x^2}{x^3+1}$

$$f'(x) = \frac{(x^3+1)(1+x^2)' - (1+x^2)(x^3+1)'}{(x^3+1)^2}$$

$$f'(x) = \frac{(x^3+1)(2x) - (1+x^2)(3x^2)}{(x^3+1)^2} = \frac{-x^4 - 3x^2 + 2x}{(x^3+1)^2}$$

b) $f(x) = (x+5)(2x-7)^{-1} = \frac{x+5}{2x-7}$

$$f'(x) = \frac{(2x-7)(x+5)' - (x+5)(2x-7)'}{(2x-7)^2}$$

$$f'(x) = \frac{(2x-7)(1) - (x+5)(2)}{(2x-7)^2} = \frac{-17}{(2x-7)^2}$$

Applications of Derivatives

Example 8: Find the equation of the tangent line to the graph of $f(x) = x^3 - 2x + 1$ at x=2.

$f(2) = 2^3 - 2(2) + 1 = 5$ pt $(2,5)$

$f'(x) = 3x^2 - 2$

$f'(2) = 3(2)^2 - 2 = 10$ slope

equation $y - y_1 = m(x - x_1)$

$y - 5 = 10(x-2)$

$y - 5 = 10x - 20$

$\boxed{y = 10x - 15}$

Example 9: Find the points (x,y) on the graph of $f(x) = -x^3 + x^2 + 5x - 1$ at which the tangent line is horizontal.

means slope is zero $\Rightarrow f'(x) = 0$

$f'(x) = -3x^2 + 2x + 5$

$0 = -3x^2 + 2x + 5$

$0 = 3x^2 - 2x - 5$

$0 = (3x - 5)(x + 1)$

$x = \frac{5}{3}$ $\quad x = -1$

points

$f(-1) = -(-1)^3 + (-1)^2 + 5(-1) - 1$
$= -4$

$f(5/3) = -(5/3)^3 + (5/3)^2 + 5(5/3) - 1$
$= \frac{148}{27}$

So the points where tangent line is horizontal are $(-1, -4)$ and $(5/3, 148/27)$

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$y^{(n)}$	"y super n"
$\frac{d^ny}{dx^n}$	"d to the n of y by dx to the n"
D^n	"d to the n"

Example 10: Find first four derivatives of $f(x) = x^3 - 3x^2 + 2$.

$f'(x) = 3x^2 - 6x$

$f''(x) = 6x - 6$

$f'''(x) = 6$

$f^{(4)}(x) = 0$