## Section 3.8 Derivatives of Inverse Functions and Logarithms

**THEOREM 3—The Derivative Rule for Inverses** If $f$ has an interval $I$ as domain and $f'(x)$ exists and is never zero on $I$, then $f^{-1}$ is differentiable at every point in its domain (the range of $f$). The value of $(f^{-1})'$ at a point $b$ in the domain of $f^{-1}$ is the reciprocal of the value of $f'$ at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))} \tag{1}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}.$$

**Example:** The function $f(x) = x^2$, x > 0 and its inverse $f^{-1}(x) = \sqrt{x}$ have derivatives $f^{-1}(x) = 2x$ and $(f^{-1})'(x) = \frac{1}{(2\sqrt{x})}$. Verify that Theorem 3 gives the same formula for the derivative of $f^{-1}(x)$:

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0. \tag{2}$$

**Example:** Find the following derivatives:

a) $\frac{d}{dx} ln\ 2x$

b) Let $u = x^2 + 3$ $\qquad \frac{d}{dx} ln\ (x^2 + 3)$

If $a > 0$ and $u$ is a differentiable function of $x$, then $a^u$ is a differentiable function of $x$ and

$$\frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}. \tag{5}$$

**Example:** Find the following derivatives:

a) $\frac{d}{dx} 3^x$ b) $\frac{d}{dx} 3^{-x}$ c) $\frac{d}{dx} 3^{sinx}$

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx} \log_a u = \frac{1}{u \ln a} \frac{du}{dx}. \tag{7}$$

Example: Find the derivative of y with respect to the given independent variable.

$y = \log_2 5\theta$

Logarithmic Differentiation. Example: Find $\frac{dy}{dx}$ = if $y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}$

## Section 3.8 Derivatives of Inverse Functions and Logarithms

**THEOREM 3—The Derivative Rule for Inverses** If $f$ has an interval $I$ as domain and $f'(x)$ exists and is never zero on $I$, then $f^{-1}$ is differentiable at every point in its domain (the range of $f$). The value of $(f^{-1})'$ at a point $b$ in the domain of $f^{-1}$ is the reciprocal of the value of $f'$ at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))} \tag{1}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}.$$

**Example:** The function $f(x) = x^2$, $x > 0$ and its inverse $f^{-1}(x) = \sqrt{x}$ have derivatives $f^{-1}(x) = 2x$ and $(f^{-1})'(x) = \frac{1}{(2\sqrt{x})}$. Verify that Theorem 3 gives the same formula for the derivative of $f^{-1}(x)$:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{2(f^{-1}(x))}$$

$$= \boxed{\frac{1}{2\sqrt{x}}} \checkmark$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \quad u > 0. \tag{2}$$

**Example:** Find the following derivatives:

a) $\frac{d}{dx} ln\ 2x$ $\quad \underline{u = 2x}$

$$= \frac{1}{2x} \cdot \frac{d}{dx}(2x)$$

$$= \frac{1}{2x} \cdot (2) = \boxed{\frac{1}{x}} \quad , x > 0$$

b) Let $u = x^2 + 3$ $\quad \frac{d}{dx} ln\ (x^2 + 3)$

$$= \frac{1}{x^2+3} \cdot \frac{d}{dx}(x^2+3) = \frac{1}{x^2+3} \cdot 2x$$

$$= \boxed{\frac{2x}{x^2+3}}$$

If $a > 0$ and $u$ is a differentiable function of $x$, then $a^u$ is a differentiable function of $x$ and

$$\frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}. \qquad (5)$$

**Example:** Find the following derivatives:

a) $\frac{d}{dx} 3^x$

$= 3^x \ln 3 \frac{d}{dx} x$

$= 3^x \ln 3 \, (1)$

$= \boxed{3^x \ln 3}$

b) $\frac{d}{dx} 3^{-x}$

$= 3^{-x} \ln 3 \frac{d}{dx}(-x)$

$= 3^{-x} \ln 3 (-1)$

$= \boxed{-3^{-x} \ln 3}$

c) $\frac{d}{dx} 3^{sinx}$

$= 3^{\sin x} \ln 3 \frac{d}{dx} \sin x$

$= \boxed{3^{\sin x} \ln 3 (\cos x)}$

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx} \log_a u = \frac{1}{u \ln a} \frac{du}{dx}. \qquad (7)$$

Example: Find the derivative of y with respect to the given independent variable.

$y = \log_2 5\theta$

$u = 5\theta$

$= \frac{1}{5\theta \ln 2} \cdot \frac{d}{d\theta}(5\theta)$

$= \frac{1}{5\theta \ln 2} \cdot 5 = \boxed{\frac{1}{\theta \ln 2}}$

Logarithmic Differentiation. Example: Find $\frac{dy}{dx}$ = if $y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}$

$\ln y = \ln \frac{(x^2+1)(x+3)^{\frac{1}{2}}}{x-1}$ (take log of both sides)

$= \ln (x^2+1)(x+3)^{\frac{1}{2}} - \ln(x-1)$ (quotient rule)

$= \ln(x^2+1) + \ln(x+3)^{\frac{1}{2}} - \ln(x-1)$ (product rule)

$= \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1)$ (power rule)

take derivatives!

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2+1} \cdot 2x + \frac{1}{2} \cdot \frac{1}{x+3} - \frac{1}{x-1}$$

solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right)$$

$$\boxed{\frac{dy}{dx} = \frac{(x^2+1)(x+3)^{\frac{1}{2}}}{x-1}\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right)}$$

The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to use the laws of logarithms to simplify the formulas before differentiating.