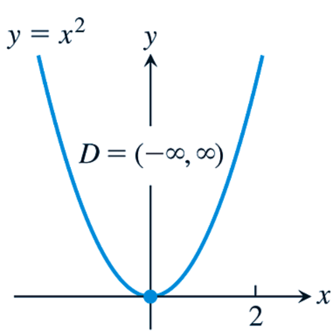
Section 4.1 Extreme Values of Functions on Closed Intervals.

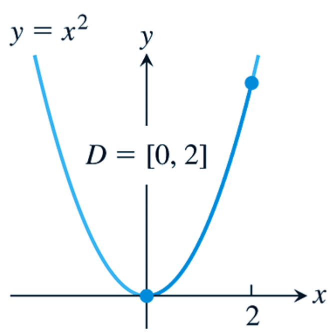
Let f be a function with domain D.
Then f has an **absolute maximum** value on D at a point c if $f(x) \le f(c)$ for all x in D and an **absolute minimum** value on D at a point c if $f(x) \ge f(c)$ for all x in D.

Example 1: The absolute extrema of the following functions on their domains can be seen in the figure below. Each function has the same defining equation, $y = x^2$, but the domains vary.

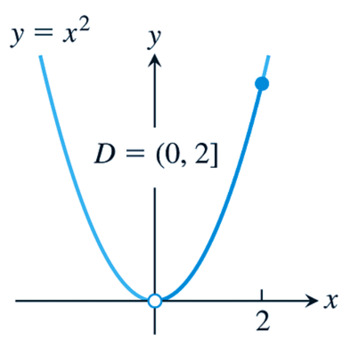
Function rule	Domain D	Absolute extrema on D
(a) $y = x^2$	$(-\infty, \infty)$	No absolute maximum. Absolute minimum of 0 at $x = 0$.
(b) $y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$. Absolute minimum of 0 at $x = 0$.
(c) $y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$. No absolute minimum.
(d) $y = x^2$	$(0, 2)$	No absolute extrema.



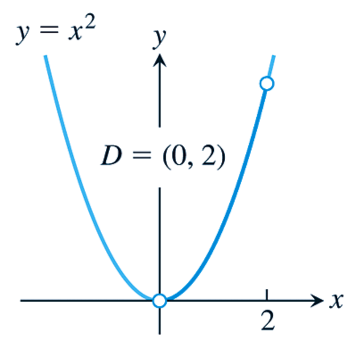
(a) abs min only



(b) abs max and min

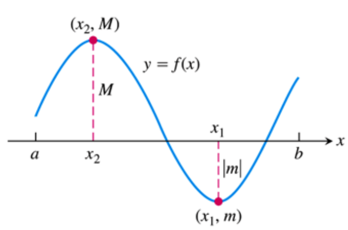


(c) abs max only

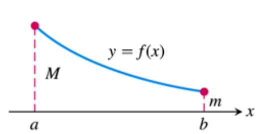


(d) no max or min

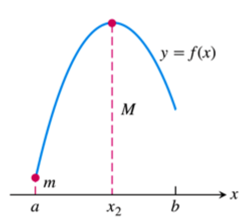
THEOREM 1—The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \le f(x) \le M$ for every other x in $[a, b]$.



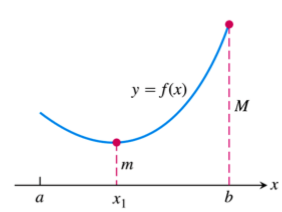
Maximum and minimum at interior points



Maximum and minimum at endpoints



Maximum at interior point, minimum at endpoint



Minimum at interior point, maximum at endpoint

DEFINITIONS A function f has a **local maximum** value at a point c within its domain D if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing c.

A function f has a **local minimum** value at a point c within its domain D if $f(x) \geq f(c)$ for all $x \in D$ lying in some open interval containing c.

Let f be a function with domain D.
Then f has an **absolute maximum** value on D at a point c if $f(x) \leq f(c)$ for all x in D and an **absolute minimum** value on D at a point c if $f(x) \geq f(c)$ for all x in D.

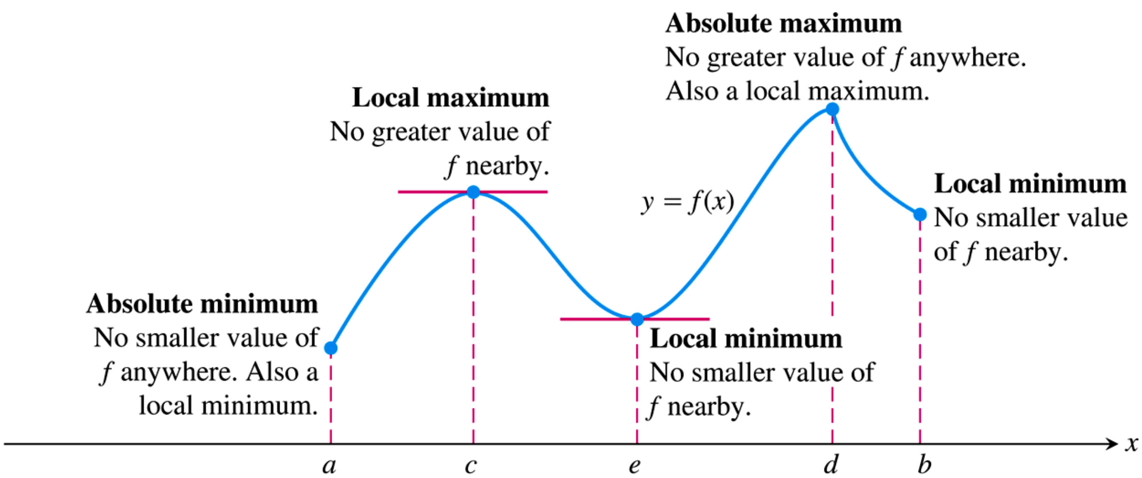


FIGURE 4.5 How to identify types of maxima and minima for a function with domain $a \leq x \leq b$.

THEOREM 2—The First Derivative Theorem for Local Extreme Values If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

DEFINITION An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

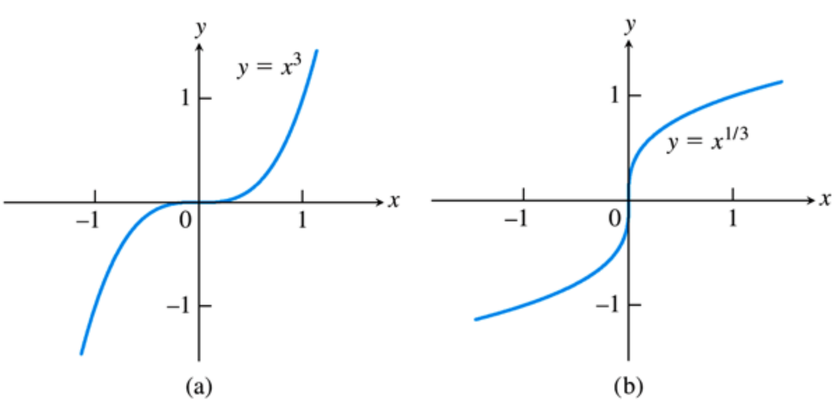


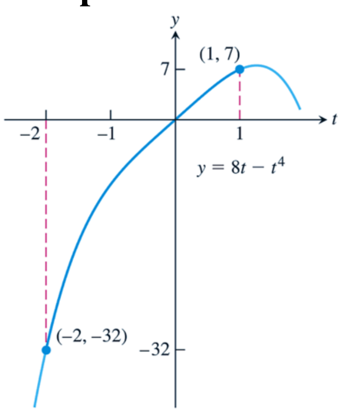
FIGURE 4.7 Critical points without extreme values. (a) $y' = 3x^2$ is 0 at $x = 0$, but $y = x^3$ has no extremum there. (b) $y' = (1/3)x^{-2/3}$ is undefined at $x = 0$, but $y = x^{1/3}$ has no extremum there.

Finding the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

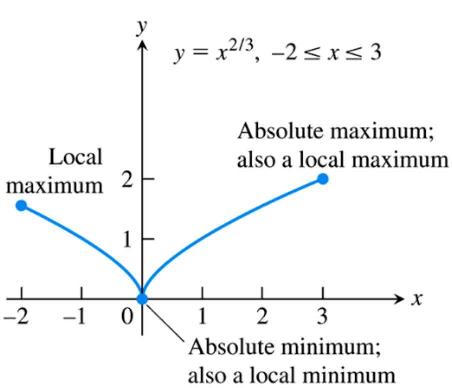
1. Find all critical points of f on the interval.
2. Evaluate f at all critical points and endpoints.
3. Take the largest and smallest of these values.

Example 2: Find the absolute maximum and minimum values of $f(x) = x^2$ on [-2, 1].

Example 3: Find the absolute maximum and minimum values of $g(t) = 8t - t^4$ on [-2, 1].



Example 4: Find the absolute maximum and minimum values of $f(x) = x^{\frac{2}{3}}$ on [-2, 3].



Finding the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Find all critical points of f on the interval.
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Example 2: Find the absolute maximum and minimum values of $f(x) = x^2$ on [-2, 1].

$f'(x) = 2x$

$0 = 2x$

$0 = x$

critical pt

$f(0) = 0^2 = 0$ MIN

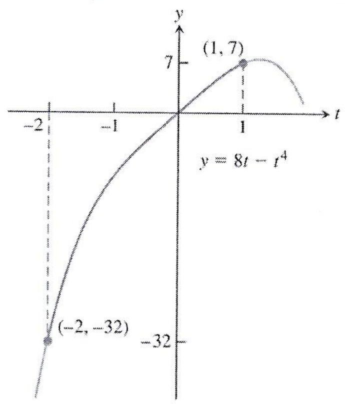
$f(-2) = (-2)^2 = 4$ Max

$f(1) = (1)^2 = 1$

Abs Max $(-2, 4)$

Abs Min $(0, 0)$

Example 3: Find the absolute maximum and minimum values of $g(t) = 8t - t^4$ on [-2, 1].



$g'(t) = 8 - 4t^3$

$0 = 8 - 4t^3$

$4t^3 = 8$

$t^3 = 2$

$t = \sqrt[3]{2}$

not in interval

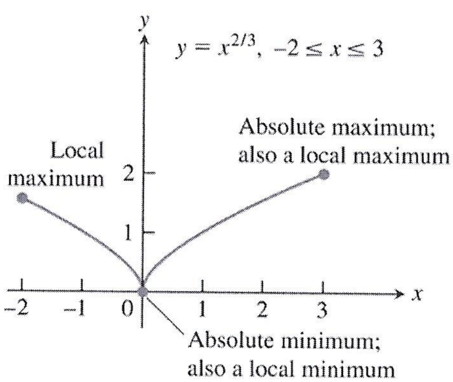
$g(-2) = -32$

$g(1) = 7$

Abs Max value of 7 when $t = 1$

Abs Min value of -32 when $t = -2$

Example 4: Find the absolute maximum and minimum values of $f(x) = x^{\frac{2}{3}}$ on [-2, 3].



$f'(x) = \frac{2}{3}x^{-1/3}$

$0 = \frac{2}{3\sqrt[3]{x}}$

critical value where $x = 0$

$f(0) = 0$ Min

$f(-2) = \sqrt[3]{4}$

$f(3) = \sqrt[3]{9}$ Max