Section 4.2 Rolle's and Mean Value Theorems

THEOREM 3—Rolle's Theorem
Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.

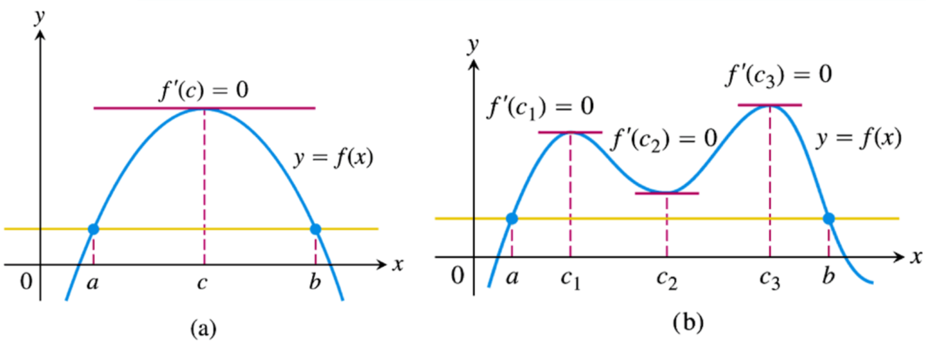

FIGURE 4.10 Rolle's Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (a), or it may have more (b).

Example 1: Verify that the conditions of Rolle's Theorem are met for $f(x) = x^3 - 3x^2 + 2x$ on the interval [0, 2] and find all values of c that satisfy the conclusion of the theorem.

Conditions: 1) continuous? Yes
2) differentiable? Yes, and $f'(x) = 3x^2 - 6x + 2$
3) f(a)=f(b)? Yes, f(0)=f(2), both are =0

Conclusion: Then there exists at least one number c in [0, 2] such that $f'(c) = 0$.

$f'(c) = 3c^2 - 6c + 2$

$0 = 3c^2 - 6c + 2$ use the quadratic formula

$c = 1 \pm \frac{\sqrt{3}}{3}$ both values of c are in the interval from [0, 2]

Example 2: Verify that the conditions of Rolle's Theorem are met for $f(x) = \sin x$ on the interval [0, 2π] and find all values of c that satisfy the conclusion of the theorem.

Conditions: 1) continuous? Yes
2) differentiable? Yes, and $f'(x) = \cos x$
3) f(a)=f(b)? Yes, f(0)=f(2π), both are =0

Conclusion: Then there exists at least one number c in [0, 2π] such that $f'(c) = 0$.

$f'(c) = \sin c$

$0 = \sin c$

$c = \frac{\pi}{2}, \frac{3\pi}{2}$

THEOREM 4—The Mean Value Theorem
Suppose $y = f(x)$ is continuous over a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \quad (1)$$

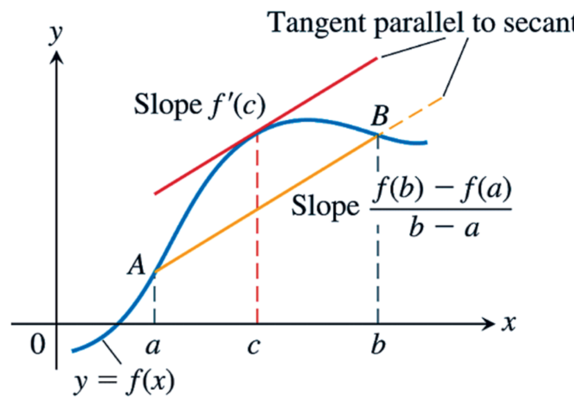

FIGURE 4.13 Geometrically, the Mean Value Theorem says that somewhere between a and b the curve has at least one tangent line parallel to the secant line that joins A and B.

Example 3: Verify that the conditions of Mean Value Theorem are met for $f(x) = x^3 + 1$ on the interval [1, 2] and find all values of c that satisfy the conclusion of the theorem.

Conditions: 1) continuous? Yes
2) differentiable? Yes, and $f'(x) = 3x^2$

Conclusion: Then there exists at least one number c in [1, 2] such that $\frac{f(b)-f(a)}{b-a} = f'(c)$.

$\frac{f(2)-f(1)}{2-1} = f'(c)$

$\frac{9-2}{2-1} = 3c^2$

$7 = 3c^2$

$\frac{7}{3} = c^2$

$\pm\sqrt{\frac{7}{3}} = c$ only $\sqrt{\frac{7}{3}} = c$ lies in the interval [1, 2]

Example 4: Verify that the conditions of Mean Value Theorem are met for $f(x) = \sqrt{25 - x^2}$ on the interval [-5, 3] and find all values of c that satisfy the conclusion of the theorem.

Conditions: 1) continuous? Yes
2) differentiable? Yes, and $f'(x) = \frac{x}{\sqrt{25-x^2}}$

Conclusion: Then there exists at least one number c in [-5, 3] such that $\frac{f(b)-f(a)}{b-a} = f'(c)$.

$\frac{f(3)-f(-5)}{3-(-5)} = f'(c)$

$\frac{4}{8} = \frac{c}{\sqrt{25-c^2}}$

$\frac{1}{2} = \frac{c}{\sqrt{25-c^2}}$

$2c = \sqrt{25 - c^2}$

$4c^2 = 25 - c^2$

$5c^2 = 25$

$c^2 = 5$

$c = \pm\sqrt{5}$, only $c = \sqrt{5}$ checks and is in the interval