Section 4.4 Concavity and Curve Sketching

A function is concave up when $f''(x) > 0$
A function is concave down when $f''(x) < 0$
A function has an inflection point when $f''(x)$ changes from + to – or from – to +

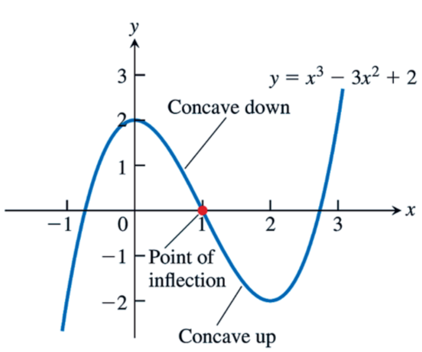
For the following examples, find the intervals where f(x) is concave up and concave down. Find inflections points and give your answer as (x, y) points. Sketch the graph of f(x).

Example 1: $f(x) = x^3 - 3x^2 + 2$
Solve $f''(x) = 0$ to find critical values

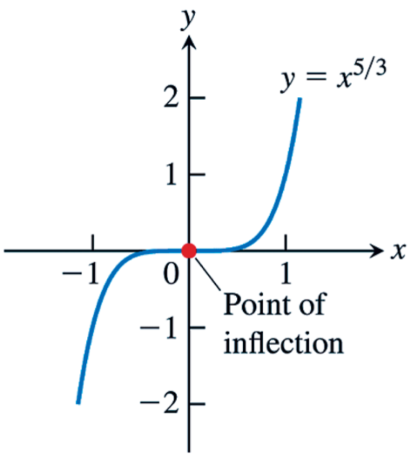
Make a second derivative number line using the critical values. Pick a test number in each interval and plug into the second derivative. Record the positive or negative sign of the results on the number line.

Using the number line, find the open intervals of the domain when $f(x)$ is concave up and concave down,

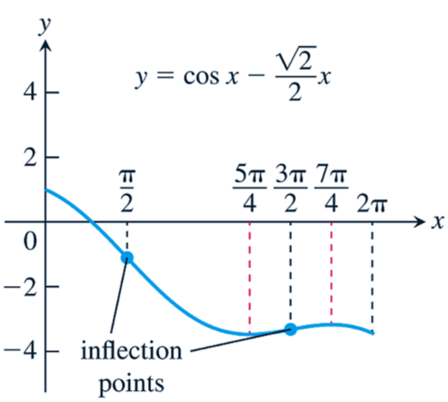
Using the number line, find the inflection points. Plug x value into $f(x)$ to find y value of points. Look at the graph and notice the concavity change at the inflection point.



Example 2: $f(x) = \sqrt[3]{x^5}$



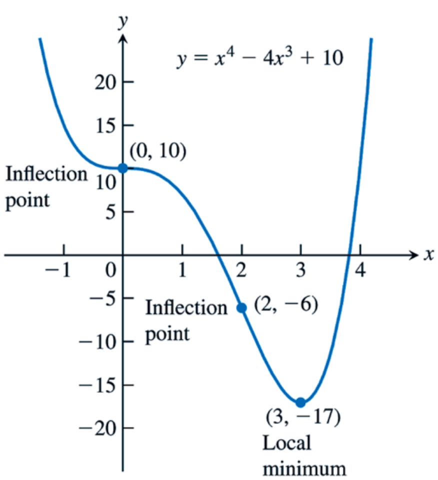
Example 3: $f(x) = \cos x - \frac{\sqrt{2}}{2}x$



Procedure for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find the derivatives y' and y''.
3. Find the critical points of f, if any, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes that may exist.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

Example 4: Sketch a graph of $f(x) = x^4 - 4x^3 + 10$. Give open intervals where f(x) is increasing, decreasing, concave up, and concave down. Give the points where f(x) has relative extrema and inflection points.



Example 5: Sketch a graph of $f(x) = x^4 - 6x^2$. Give open intervals where f(x) is increasing, decreasing, concave up, and concave down. Give the points where f(x) has relative extrema and inflection points.

Example 6: Sketch a graph of $f(x) = 4x^3 - 48x$. Give open intervals where f(x) is increasing, decreasing, concave up, and concave down. Give the points where f(x) has relative extrema and inflection points.

Section 4.4 Concavity and Curve Sketching

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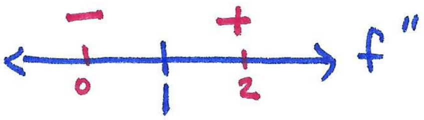
For the following examples, find the intervals where f(x) is concave up and concave down. Find inflections points and give your answer as (x, y) points. Sketch the graph of f(x).

Example 1: $f(x) = x^3 - 3x^2 + 2$
Solve $f''(x) = 0$ to find critical values

$f(x) = x^3 - 3x^2 + 2$
$f'(x) = 3x^2 - 6x$
$f''(x) = 6x - 6$

$f''(x) = 6x - 6 = 0$
$6x = 6$
$x = 1$

Make a second derivative number line using the critical values. Pick a test number in each interval and plug into the second derivative. Record the positive or negative sign of the results on the number line.

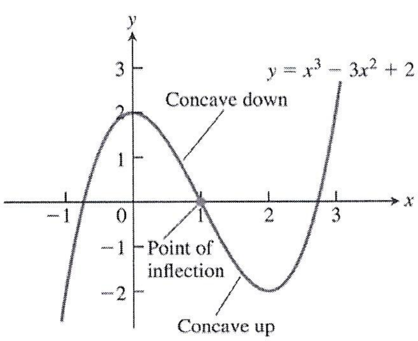


$f''(0) = 6(0) - 6 = -$
$f''(2) = 6(2) - 6 = +$

Using the number line, find the open intervals of the domain when $f(x)$ is concave up and concave down,

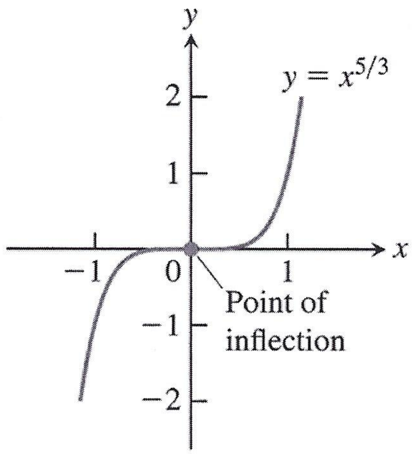
Concave up $(1, \infty)$
Concave down $(-\infty, 1)$

Using the number line, find the inflection points. Plug x value into $f(x)$ to find y value of points. Look at the graph and notice the concavity change at the inflection point.



$(1, 0)$ is the inflection point
$f(1) = 1^3 - 3(1)^2 + 2 = 0$

Example 2: $f(x) = \sqrt[3]{x^5}$

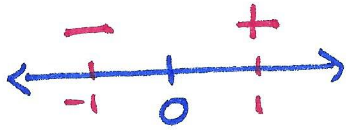


$f(x) = x^{5/3}$

$f'(x) = \frac{5}{3} x^{2/3}$

$f''(x) = \frac{10}{9} x^{-1/3} = \frac{10}{9\sqrt[3]{x}}$

$x = 0$ critical value



$f''(-1) = \frac{10}{9\sqrt[3]{-1}} = -$

$f''(1) = \frac{10}{9\sqrt[3]{1}} = +$

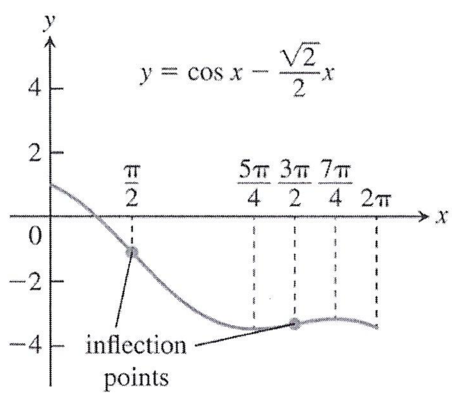
concave up $(0, \infty)$

concave down $(-\infty, 0)$

Inflection point at $(0, 0)$ $f(0) = \sqrt[3]{0^5} = 0$

on $[0, 2\pi]$

Example 3: $f(x) = \cos x - \frac{\sqrt{2}}{2} x$



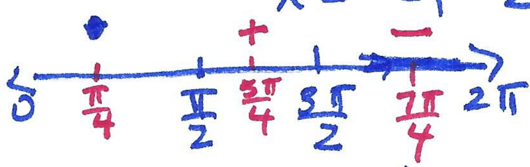
$f(x) = \cos x - \frac{\sqrt{2}}{2} x$

$f'(x) = -\sin x - \frac{\sqrt{2}}{2}$

$f''(x) = -\cos x = 0$

$\cos x = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$



concave up $(\frac{\pi}{2}, \frac{3\pi}{2})$

concave down $(0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$

inflection points

$(\frac{\pi}{2}, -1.11)$

$(\frac{3\pi}{2}, -3.33)$

* use calculator $f(\pi/2) = -1.11$

$f(3\pi/2) = -3.33$

Procedure for Graphing $y = f(x)$

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Example 4: Sketch a graph of $f(x) = x^4 - 4x^3 + 10$. Give open intervals where f(x) is increasing, decreasing, concave up, and concave down. Give the points where f(x) has relative extrema and inflection points.

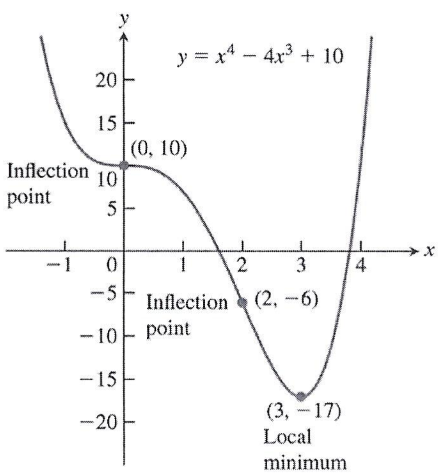
First Derivative

$f'(x) = 4x^3 - 12x^2 = 0$

$4x^2(x-3) = 0$

$x = 0 \quad x = 3$

f': $-$ | 0 | $-$ | 3 | $+$



Increasing $(3, \infty)$

Decreasing $(-\infty, 0) \cup (0, 3)$

Rel or Local Max None

Relative or Local Min $(3, -17)$

Second Derivative

$f''(x) = 12x^2 - 24x = 12x(x-2) = 0$

$x = 0 \quad x = 2$

f'': $+$ | 0 | $-$ | 2 | $+$

Concave Up $(-\infty, 0) \cup (2, \infty)$

Concave Down $(0, 2)$

Inflection Points $(0, 10)$ and $(2, -6)$

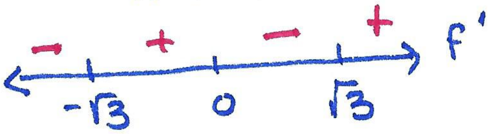
Example 5: Sketch a graph of $f(x) = x^4 - 6x^2$. Give open intervals where f(x) is increasing, decreasing, concave up, and concave down. Give the points where f(x) has relative extrema and inflection points.

First Derivative

$f'(x) = 4x^3 - 12x = 0$

$4x(x^2 - 3) = 0$

$x = 0 \quad x = \pm\sqrt{3}$

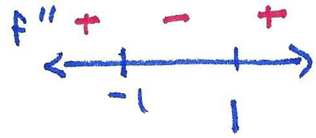


Second Derivative

$f''(x) = 12x^2 - 12 = 0$

$12(x^2 - 1) = 0$

$x = 1 \quad x = -1$



Concave Up $(-\infty, -1) \cup (1, \infty)$

Concave Down $(-1, 1)$

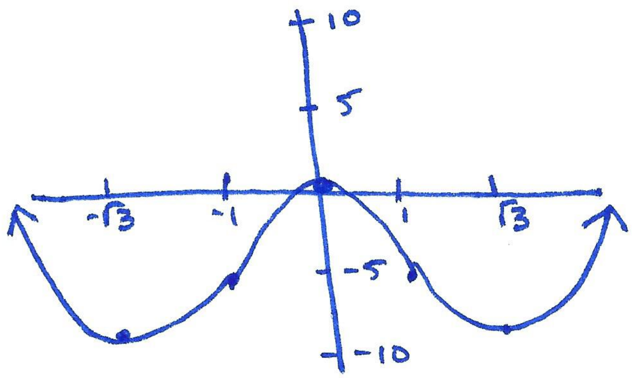
Inflection Points $(-1, -5)$ and $(1, -5)$

Increasing $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$

Decreasing $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$

Local Max $(0, 0)$

Local Mins $(-\sqrt{3}, -9)$
$(\sqrt{3}, 9)$



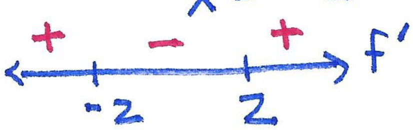
Example 6: Sketch a graph of $f(x) = 4x^3 - 48x$. Give open intervals where f(x) is increasing, decreasing, concave up, and concave down. Give the points where f(x) has relative extrema and inflection points.

First Derivative

$f'(x) = 12x^2 - 48 = 0$

$12(x^2 - 4) = 0$

$x = \pm 2$



Increasing $(-\infty, -2) \cup (2, \infty)$

Decreasing $(-2, 2)$

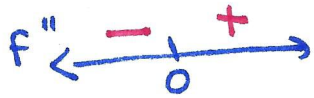
Local Max $(-2, 64)$

Local Min $(2, -64)$

Second Derivative

$f''(x) = 24x = 0$

$x = 0$



Concave Up $(-\infty, 0)$

Concave Down $(0, \infty)$

Inflection Point $(0, 0)$