Section 4.5 Indeterminate Forms and L'Hopital's Rule

L'Hôpital's Rule enables us to draw on our success with derivatives to evaluate limits that otherwise lead to indeterminate forms such as $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty \cdot 0$, and $\infty - \infty$.

THEOREM 6—L'Hôpital's Rule Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

Example: The following limits involve 0/0 indeterminate forms, so we apply L'Hôpital's Rule. In some cases, it must be applied repeatedly.

$a)$ $\lim\limits_{x \to 0} \frac{3x - \sin x}{x}$

$b)$ $\lim\limits_{x \to 0} \frac{\sqrt{1+x} - 1}{x}$

c) $\lim\limits_{x \to 0} \frac{x - \sin x}{x^3}$

Using L’Hôpital’s Rule

To find

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

by l’Hôpital’s Rule, we continue to differentiate f and g, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L’Hôpital’s Rule does not apply when either the numerator or denominator has a finite nonzero limit.

Example: L'Hôpital's Rule applies to one-sided limits as well.

a) $\lim\limits_{x \to 0+} \frac{\sin x}{x^2}$

b) $\lim\limits_{x \to 0-} \frac{\sin x}{x^2}$

Example: Find the limits of these ∞/∞ forms.

a) $\lim\limits_{x \to \pi/2} \frac{\sec x}{1+\tan x}$

b) $\lim\limits_{x \to \infty} \frac{\ln x}{2\sqrt{x}}$

c) $\lim\limits_{x \to \infty} \frac{e^x}{x^2}$

Example: Find the limits of these $\infty \cdot 0$ forms.

a) $\lim\limits_{x \to 0+} \sqrt{x} \ln x$

Example: Find the limit of this $\infty - \infty$ form.

a) $\lim\limits_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$

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> **THEOREM 6—L'Hôpital's Rule** Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then
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> $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$
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> assuming that the limit on the right side of this equation exists.

Example: The following limits involve 0/0 indeterminate forms, so we apply L'Hôpital's Rule. In some cases, it must be applied repeatedly.

a) $\lim\limits_{x\to 0} \frac{3x - \sin x}{x}$ $= \lim\limits_{x\to 0} \frac{3-\cos x}{1}$

$= \frac{3-\cos x}{1}\Big|_{x=0} = \frac{3-1}{1} = 2$

b) $\lim\limits_{x\to 0} \frac{\sqrt{1+x}-1}{x}$ $= \lim\limits_{x\to 0} \frac{(1+x)^{\frac{1}{2}}-1}{x}$

$= \lim\limits_{x\to 0} \frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}}{1}$

$= \lim\limits_{x\to 0} \frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{\frac{1}{2\sqrt{1+0}}}{1} = \frac{1}{2}$

c) $\lim\limits_{x\to 0} \frac{x - \sin x}{x^3}$ $\frac{0}{0}$ DO

$= \lim\limits_{x\to 0} \frac{1-\cos x}{3x^2}$ still $\frac{0}{0}$ continue

$= \lim\limits_{x\to 0} \frac{\sin x}{6x}$ still $\frac{0}{0}$ continue

$= \lim\limits_{x\to 0} \frac{\cos x}{6} = \frac{\cos(0)}{6} = \frac{1}{6}$

Using L'Hôpital's Rule

To find

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, we continue to differentiate f and g, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

Example: L'Hôpital's Rule applies to one-sided limits as well.

a) $\lim\limits_{x \to 0+} \frac{\sin x}{x^2} = \lim\limits_{x \to 0+} \frac{\cos x}{2x} = +\infty$

b) $\lim\limits_{x \to 0-} \frac{\sin x}{x^2} = \lim\limits_{x \to 0^-} \frac{\cos x}{2x} = -\infty$

Example: Find the limits of these ∞/∞ forms.

a) $\lim\limits_{x \to \pi/2} \frac{\sec x}{1+\tan x}$

discontinuous at $\frac{\pi}{2}$ so look at one-sided limits.

$\lim\limits_{x \to \pi/2^-} \frac{\sec x}{1+\tan x}$ $\frac{\infty}{\infty}$ ∞

$\lim\limits_{x \to \pi/2^-} \frac{\sec x \tan x}{\sec^2 x} =$

$\lim\limits_{x \to \pi/2^-} \sin x = 1$

also $\lim\limits_{x \to \pi/2^+} \sin x = 1$ so two sided limit $= 1$

b) $\lim\limits_{x \to \infty} \frac{\ln x}{2\sqrt{x}} = \lim\limits_{x \to \infty} \frac{\ln x}{2(x)^{\frac{1}{2}}}$

$= \lim\limits_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}}$

$= \lim\limits_{x \to \infty} \frac{1}{\sqrt{x}} = 0$

note: $\frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}} = \frac{\sqrt{x}}{x} = \frac{x^{\frac{1}{2}}}{x^{\frac{2}{2}}} = \frac{1}{\sqrt{x}}$

c) $\lim\limits_{x \to \infty} \frac{e^x}{x^2} = \lim\limits_{x \to \infty} \frac{e^x}{2x}$

$= \lim\limits_{x \to \infty} \frac{e^x}{2} = \infty$

Example: Find the limits of these $\infty \cdot 0$ forms.

a) $\lim\limits_{x\to 0+} \sqrt{x}\ln x$

note $\frac{1}{\frac{1}{\sqrt{x}}} = \sqrt{x}$

$$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2x^{3/2}}}$$

$$= \lim_{x\to 0^+} (-2\sqrt{x}) = 0$$

$$\frac{\frac{1}{x}}{-\frac{1}{2x^{3/2}}} = \frac{1}{x} \cdot \frac{-2x^{\frac{3}{2}}}{1}$$

$$= \frac{-2x^{\frac{3}{2}}}{x^{\frac{2}{2}}} = -2x^{\frac{1}{2}}$$

Example: Find the limit of this $\infty - \infty$ form.

a) $\lim\limits_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$

need to combine fractions first!

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{1}{\sin x}\cdot\frac{x}{x} - \frac{1}{x}\cdot\frac{\sin x}{\sin x}$$

$$= \frac{x - \sin x}{x\sin x}$$

$$= \lim_{x\to 0}\left(\frac{x-\sin x}{x\sin x}\right) \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x} \quad \left(\frac{0}{0}\right) \text{ so}$$

$$= \lim_{x\to 0} \frac{\sin x}{\cos x + \cos x - x\sin x}$$

$$= \lim_{x\to 0} \frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0$$