# 5.4

## Mean Value Theorem for Definite Integrals

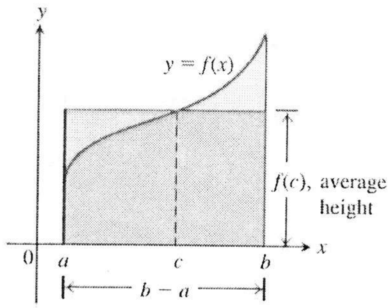


FIGURE 5.16 The value $f(c)$ in the Mean Value Theorem is, in a sense, the average (or *mean*) height of $f$ on $[a, b]$. When $f \geq 0$, the area of the rectangle is the area under the graph of $f$ from $a$ to $b$,

$$f(c)(b-a) = \int_a^b f(x)\,dx.$$

If $f$ is continuous on $[a,b]$, then at some point $c$ in $[a,b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

**Example** Find the average value of $f(x) = \sqrt{4-x^2}$ on $[-2, 2]$ then find value of c.

$$f(c) = \frac{1}{2-(-2)}\int_{-2}^{2} \sqrt{4-x^2}\,dx$$

$$= \frac{1}{4}(\text{Area semi circle})$$

$$= \frac{1}{4}\left(\frac{1}{2}\pi(2)^2\right)$$

$$f(c) = \frac{\pi}{2}$$ ← Ave value

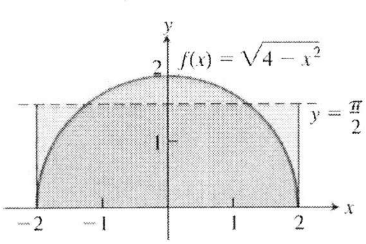


FIGURE 5.15 The average value of $f(x) = \sqrt{4-x^2}$ on $[-2, 2]$ is $\pi/2$ (Example 5). The area of the rectangle shown here is $4 \cdot (\pi/2) = 2\pi$, which is also the area of the semicircle.

$$\sqrt{4-c^2} = \pi/2$$

$$4-c^2 = \pi^2/4$$

$$4-\frac{\pi^2}{4} = c^2$$

$$\frac{16-\pi^2}{4} = c^2$$

$$c = \pm\sqrt{\frac{16-\pi^2}{4}}$$

$$c = \pm 1.56$$

# Fundamental Theorem of Calculus part I

If $f$ is continuous on $[a,b]$, then $F(x) = \int_a^x f(t)\,dt$
is continuous on $[a,b]$ and differentiable on $(a,b)$
and its derivative is $f(x)$.

$$F'(x) = \frac{d}{dx} \int_a^x f(t)\,dt = f(x)$$

## Examples  Find $dy/dx$

① $y = \int_a^x (t^3+1)\,dt$

$$\frac{dy}{dx} = \frac{d}{dx} \int_a^x (t^3+1)\,dt = x^3+1$$

② $y = \int_x^5 3t \sin t \, dt$

$$\frac{dy}{dx} = \frac{d}{dx} \int_x^5 3t \sin t \, dt = \frac{d}{dx}\left(-\int_5^x 3t \sin t \, dt\right)$$

$$= -3x \sin x$$

③ $$y = \int_1^{x^2} \cos t \, dt$$ ← problem

$$\frac{dy}{dx} = \frac{d}{dx} \int_1^{x^2} \cos t \, dt$$

$$\frac{dy}{dx} = \cos x^2 \cdot 2x$$

$$\frac{dy}{dx} = 2x \cos x^2$$

④ $$y = \int_{1+3x^2}^{4} \frac{1}{2+t} \, dt$$

$$\frac{dy}{dx} = \frac{d}{dx}\left( - \int_4^{1+3x^2} \frac{1}{2+t} \, dt \right)$$

$$\frac{dy}{dx} = -\left( \frac{1}{2+(1+3x^2)} \right) \cdot 6x$$

$$= \frac{-6x}{3+3x^2} = \frac{-6x}{3(1+x^2)}$$

$$= \frac{-2x}{1+x^2}$$

# Fundamental Theorem of Calculus – part II

If $f$ is continuous over $[a, b]$ and $F$ is any antiderivative of $f$, then $\int_a^b f(x)\,dx = F(b) - F(a)$

* don't need $+C$

## Examples

① $\int_0^2 x^2\,dx = \frac{1}{3}x^3 \Big|_0^2 = \frac{1}{3}(2)^3 - \frac{1}{3}(0)^3 = \frac{8}{3}$

② $\int_0^\pi \cos x\,dx = \sin x \Big|_0^\pi = \sin\pi - \sin 0 = 0 - 0 = 0$

③ $\int_{-1}^1 (x^2 - 2x + 3)\,dx = \frac{1}{3}x^3 - x^2 + 3x \Big|_{-1}^1$

$= \left(\frac{1}{3} - 1 + 3\right) - \left(-1/3 - 1 - 3\right)$

$= \frac{2}{3} + 6 = 20/3$

④ $\int_{-\pi/4}^0 \sec x \tan x\,dx = \sec x \Big|_{-\pi/4}^0 = \sec(0) - \sec(-\pi/4)$

$= 1 - \sqrt{2}$

⑤ $\int_1^4 \frac{3}{2}\sqrt{x} - \frac{4}{x^2}\,dx = x^{3/2} + \frac{4}{x} \Big|_1^4 = (8+1) - (1+4)$

$= 4$

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**Total Area**

To find the area between the graph of y=f(x) and the x-axis over the interval [a, b]:

i. Subdivide [a, b] at the zeros of f

ii. Integrate f over each subinterval

iii. Add the absolute values of the integrals

Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \leq x \leq 2$.

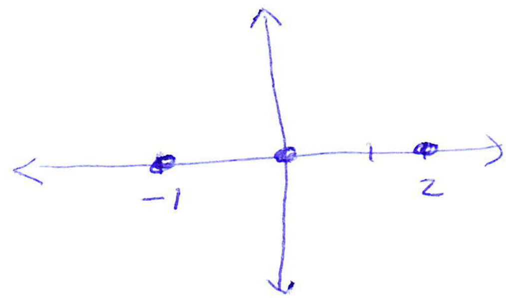
subdivide at zeros

$$f(x) = x^3 - x^2 - 2x = 0$$

$$x(x^2 - x - 2) = 0$$

$$x(x-2)(x+1) = 0$$

$$x = 0 \quad x = 2 \quad x = -1$$




integrate f over each subinterval

$$\int_{-1}^{0} x^3 - x^2 - 2x \, dx = \frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 \Big|_{-1}^{0} = 0 - (1/4 + 1/3 - 1)$$

$$= -\left(\frac{3}{12} + \frac{4}{12} - \frac{12}{12}\right) = \frac{5}{12}$$

$$\int_{0}^{2} x^3 - x^2 - 2x \, dx = \frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 \Big|_{0}^{2} = \left(4 - \frac{8}{3} - 4\right) - 0$$

$$= -8/3$$

add asolute values of the integrals

$$A = |5/12| + |-8/3| = \frac{5}{12} + \frac{32}{12} = \frac{37}{12}$$