5.6 U-sub with definite integrals

Ex $\int_{-1}^{1} 3x^2 \sqrt{x^3+1}\, dx$

$u = x^3+1$

$\frac{du}{dx} = 3x^2$

$du = 3x^2 dx$

$\int_{-1}^{1} (x^3+1)^{1/2}\, 3x^2 dx$

limits

when $x=-1$ $u=0$

when $x=1$ $u=2$

$\int_{0}^{2} u^{1/2} du$

$\frac{2}{3} u^{3/2} \Big|_0^2 = \frac{2}{3}(2)^{3/2} - \frac{2}{3}(0)^{3/2}$

$= \frac{2}{3}\sqrt[2]{8} = \frac{2}{3}\sqrt[2]{4\cdot 2}$ $\boxed{= \frac{4}{3}\sqrt{2}}$

Ex $\int_{0}^{1} \frac{4x^3}{(x^4+1)^2} dx$

$u = x^4+1$

$\frac{du}{dx} = 4x^3$

$du = 4x^3 dx$

limits

when $x=0$ $u=1$

$x=1$ $u=2$

$= \int_{1}^{2} \frac{1}{u^2} du = \int_{1}^{2} u^{-2} du$

$= \frac{1}{-1} u^{-1} \Big|_1^2 = \frac{-1}{u} \Big|_1^2$

$= -\frac{1}{2} - \left(\frac{-1}{1}\right) \boxed{= \frac{1}{2}}$

Ex $\int_0^1 x\sqrt{1-x^2}\,dx$

$u = 1-x^2$

$\frac{du}{dx} = -2x$

$du = -2x\,dx$

$-\frac{1}{2}\,du = x\,dx$

$\int_0^1 (1-x^2)^{1/2}\,x\,dx$

$\int_1^0 u^{1/2} \cdot -1/2\,du$

limits

when $x=0$ then $u=1$

$x=1$ then $u=0$

$-1/2 \int_1^0 u^{1/2}\,du$

$-1/2 \cdot \frac{2}{3} u^{3/2} \Big|_1^0 =$

$-\frac{1}{3} u^{3/2} \Big|_1^0 = -1/3(0)^{3/2} - \left(-1/3\,(1)^{3/2}\right)$

$= 0 + 1/3 \boxed{= 1/3}$

Ex $\int_1^4 \frac{dy}{2\sqrt{y}\,(1+\sqrt{y})^2}$

$u = 1+\sqrt{y} = 1+y^{1/2}$

$\frac{du}{dy} = 1/2\,y^{-1/2}$

$du = \frac{1}{2\sqrt{y}}\,dy$

$\int_2^3 \frac{1}{u^2}\,du = \int_2^3 u^{-2}\,du$

limits

when $x=1$ $u=2$

$x=4$ $u=3$

$= \frac{1}{-1} u^{-1} \Big|_2^3 = -\frac{1}{u} \Big|_2^3$

$= -\frac{1}{3} - (-1/2) \boxed{= \frac{1}{6}}$

Ex $\int_{-\sqrt{7}}^{0} t\,(t^2+1)^{1/3}\,dt$

$u = t^2+1$

$\frac{du}{dt} = 2t$

$du = 2t\,dt$

$\frac{1}{2}du = t\,dt$

$\int_{-\sqrt{7}}^{0} (t^2+1)^{1/3}\,t\,dt$

limits

when $t = -\sqrt{7}$ $u = 8$

$t = 0$ $u = 1$

$\int_{8}^{1} u^{1/3} \cdot \frac{1}{2}\,du$

$\frac{1}{2}\int_{8}^{1} u^{1/3}\,du$

$\frac{1}{2}\,\frac{3}{4}\,u^{4/3}\Big|_{8}^{1}$

$\frac{3}{8}u^{4/3}\Big|_{8}^{1} = \frac{3}{8}\cdot 1^{4/3} - \frac{3}{8}\cdot 8^{4/3}$

$= \frac{3}{8} - \frac{3}{8}(16)$

$\boxed{= \frac{-45}{8}}$

Ex $\int_{0}^{3} \sqrt{y+1}\,dy$

$u = y+1$

$\frac{du}{dy} = 1$

$du = dy$

limits

when $y = 0$ $u = 1$

$y = 3$ $u = 4$

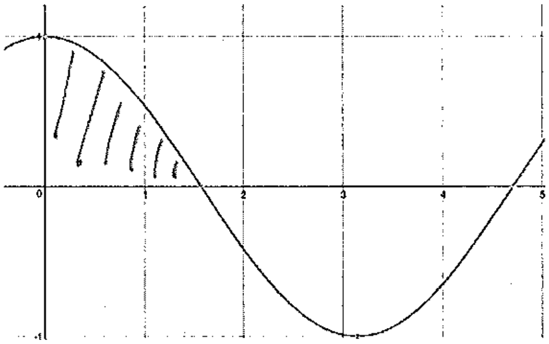
$\int_{1}^{4} u^{1/2}\,du$

$\frac{2}{3}u^{3/2}\Big|_{1}^{4}$

$\frac{2}{3}\cdot 4^{3/2} - \frac{2}{3}(1)^{3/2}$

$\frac{16}{3} - \frac{2}{3} = \frac{14}{3}$

Find the total area between the graph *f* and the x-axis.

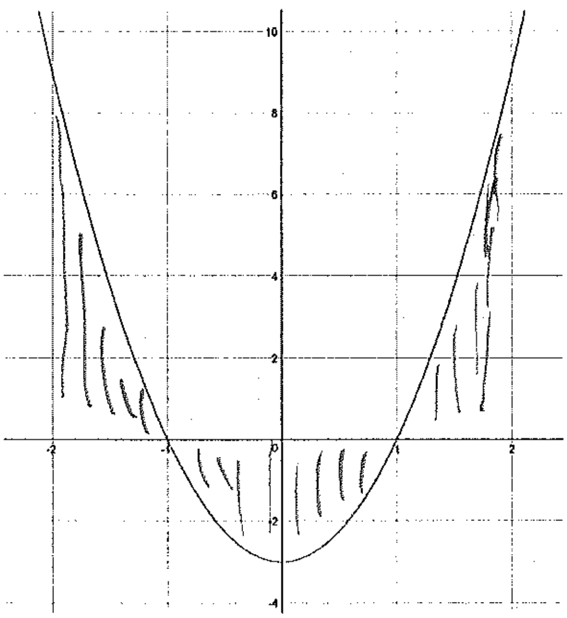


$f(x) = \cos x \qquad 0 \le x \le \pi/2$

$A = \int_0^{\pi/2} \cos x \, dx$

$= \sin x \Big|_0^{\pi/2}$

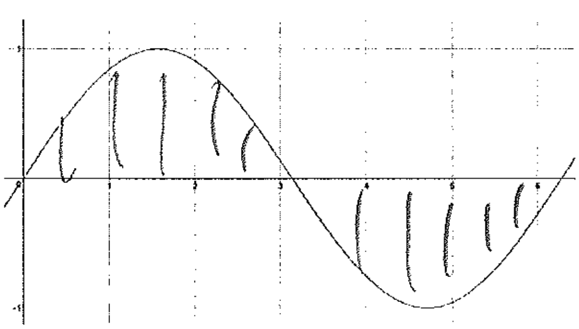
$\boxed{A = 1}$



$f(x) = 3x^2 - 3 \qquad -2 \le x \le 2$

$A = \left| \int_{-2}^{-1} 3x^2 - 3dx \right| + \left| \int_{-1}^{1} 3x^2 - 3dx \right| + \left| \int_{1}^{2} 3x^2 - 3dx \right|$

$\boxed{A = 12}$



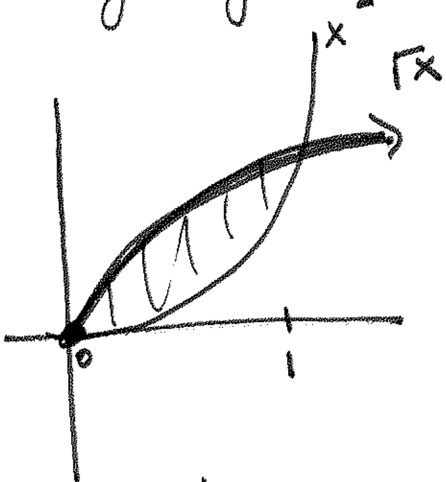
$f(x) = \sin x \qquad 0 \le x \le 2\pi$

$A = \left| \int_0^{\pi} \sin x dx \right| + \left| \int_{\pi}^{2\pi} \sin x dx \right|$

$\boxed{A = 4}$

Area between Curves

Ex Find the area of the region enclosed by $y = x^2$ and $y = \sqrt{x}$



$$\text{Area} = \int_a^b \text{top} - \text{bottom}\, dx$$

a + b are where curves intersect

set functions equal + solve

$$\sqrt{x} = x^2$$
$$(\sqrt{x})^2 = (x^2)^2$$
$$x = x^4$$
$$0 = x^4 - x$$
$$0 = x(x^3 - 1)$$
$$x = 0 \qquad x = 1$$

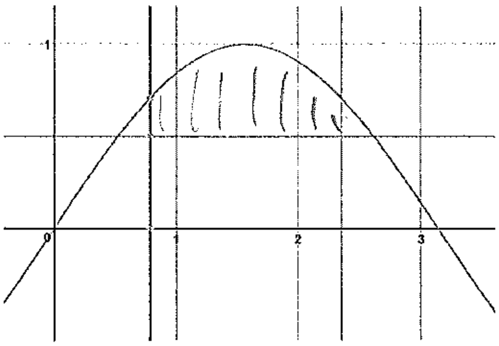
$$\text{Area} = \int_0^1 (\sqrt{x} - x^2)\, dx$$

$$\frac{2}{3}x^{3/2} - \frac{1}{3}x^3 \Big|_0^1$$

$$\left(\frac{2}{3}(1)^{3/2} - \frac{1}{3}(1)^3\right) - (0)$$

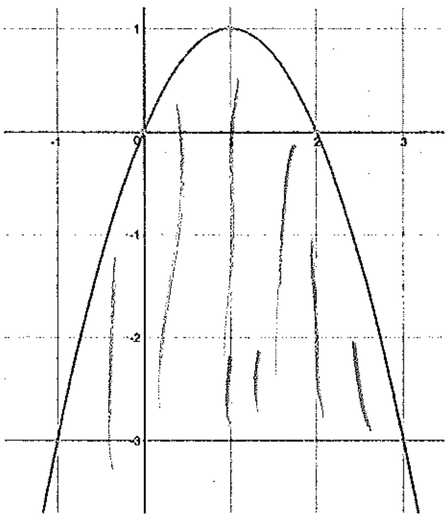
$$\frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

Find the areas of the regions enclosed by the curves and/or lines



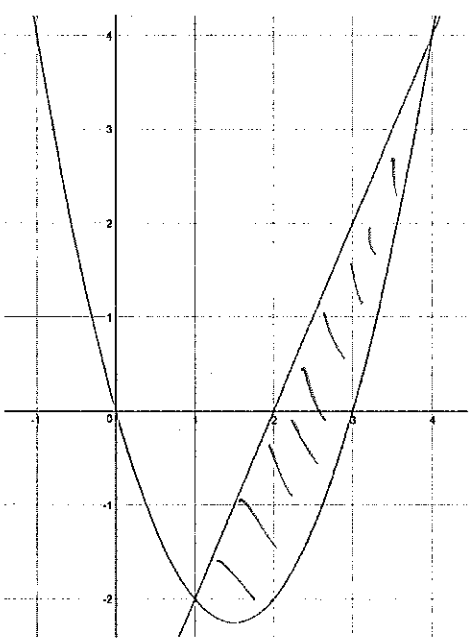
$y = \sin x$, $y = 1/2$, $x = \pi/4$, $x = 3\pi/4$

$$A = \int_{\pi/4}^{3\pi/4} \left(\sin x - \frac{1}{2}\right) dx$$



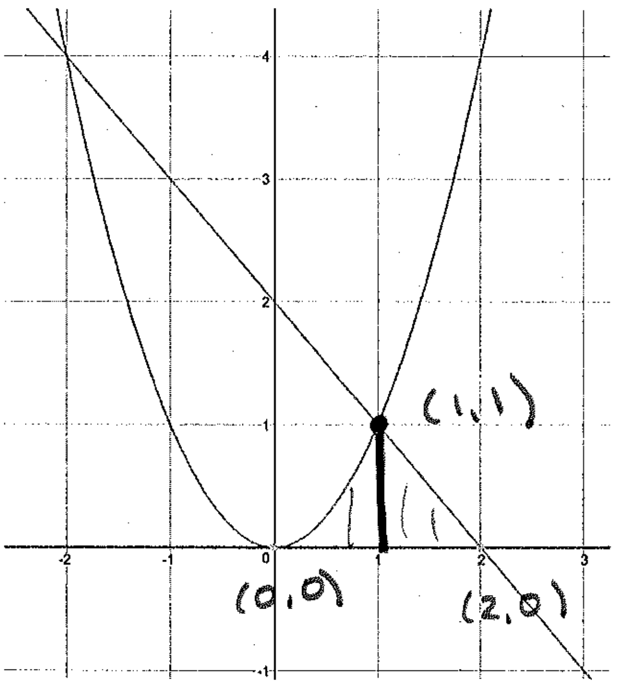
$y = 2x - x^2$, $y = -3$

$$A = \int_{-1}^{3} (2x - x^2) - (-3)\, dx$$



$y = x^2 - 3x$, $y = 2x - 4$

$$A = \int_{1}^{4} (2x-4) - (x^2 - 3x)\, dx$$

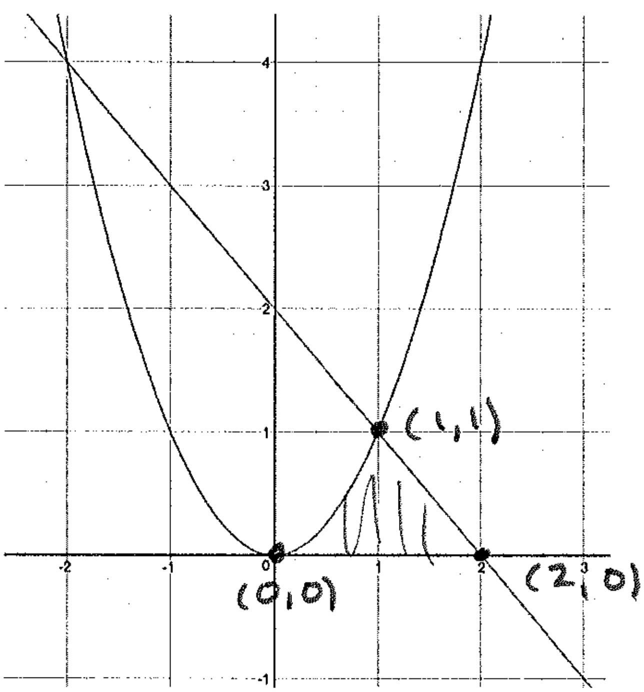


$y = 2 - x$

$y = x^2$, $x + y = 2$, $y = 0$

$\int_0^1 x^2\,dx + \int_1^2 2 - x\,dx$

$\boxed{A = 5/6}$



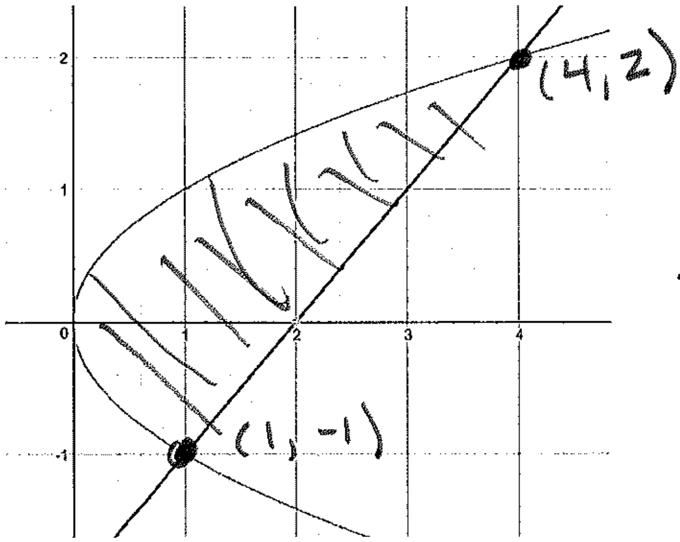
$y = x^2$, $x + y = 2$, $y = 0$

$\sqrt{y} = x$ $\quad x = 2 - y$

$\int$ right − left dy

$\int_0^1 (2 - y) - (\sqrt{y})\,dy$

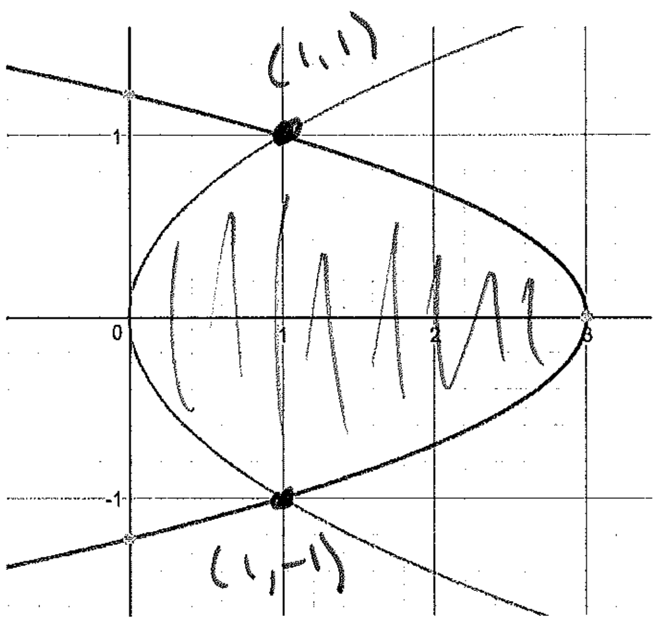
$\boxed{A = 5/6}$



$x = y^2, \quad x = y + 2$

$$\int_{-1}^{2} (y+2) - y^2 \, dy$$

$$\boxed{A = 9/2}$$



$x - y^2 = 0 \qquad x + 2y^2 = 3$

$x = y^2 \qquad x = 3 - 2y^2$

$$\int_{-1}^{1} (3 - 2y^2) - y^2 \, dy$$

$$\boxed{A = 4}$$

Evaluate the following integrals using properties or geometric formulas.

1. $\int_3^3 \sin(x^2)\,dx$ $= 0$

2. $\int_0^4 \sqrt{16-x^2}\,dx =$ $\frac{1}{4}\pi r^2 = 4\pi$

3. $\int_0^4 2x-1\,dx$ $=12$

4. Given that f is integrable on [1, 5] where $\int_1^3 f(x)\,dx = 7$ and $\int_1^5 f(x)\,dx = 16$.

Find the following. a) $\int_3^5 f(x)\,dx = 9$ b) $\int_5^1 f(x)\,dx = -16$

5. Evaluate the sum $\sum_{k=1}^{9}(-4k)$ $= -180$

Evaluate the following integrals. Use C as the arbitrary constant.

6. $\int 5x^3 + 3x^2 - 13\,dx$ $= \frac{5}{4}x^4 + x^3 - 13x + C$

7. $\int 3\cos x - 2\sin x\,dx =$ $3\sin x + 2\cos x + C$

8. $\int 3\sqrt[3]{x^2} - 2\sqrt{x^3}\,dx$
$= \frac{9}{5}x^{5/3} - \frac{4}{5}x^{5/2} + C$

9. $\int \frac{5+4x^3}{x^2}\,dx$ $= -\frac{5}{x} + 2x^2 + C$

Evaluate the following integrals

10. $\int_0^1 4x^3 - 2x^2 + x - 4\,dx$ $= -13/6$

11. $\int_1^2 2x^2(x^3+3)\,dx$ $= 35$

12. $\int_0^{\pi/6} 2\sin x - \cos x\,dx$ $\frac{3-2\sqrt{3}}{2}$

13. $\int_1^4 2x^2\sqrt{x} - \frac{1}{x^3}\,dx$ $= \frac{16151}{224}$

14. Find the total area of between the curve y=cos x and the x-axis from $0 \le x \le 2\pi$ $A_{total} = 4$

15. Find the area enclosed by $y = 4-(x-1)^2$ and $y = 0$

$\int_{-1}^{3} 4 - (x-1)^2\,dx = \int_{-1}^{3}(-x^2 + 2x + 3)\,dx = -\frac{1}{3}x^3 + x^2 + 3x \Big|_{-1}^{3}$

$= \frac{32}{3}$

Find the following integrals.

16. $\int x^2 \cos(x^3)\,dx$

$\frac{1}{3}\sin(x^3) + C$

17. $\int (4x+1)\sqrt{2x^2+x+1}\,dx$

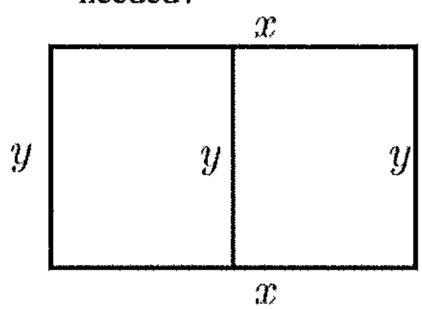
$\frac{2}{3}(2x^2+x+1)^{3/2} + C$

18. $\int \frac{\sec^2(\sqrt{x})}{\sqrt{x}}dx$

$2\tan(\sqrt{x}) + C$

19. $\int_1^5 \sqrt{5x+11}\,dx$ $= 304/15$

20. A garden with area of 216 square meters is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides (see diagram). What dimensions for the outer rectangle will require the smallest length of fence? How much fence will be needed?



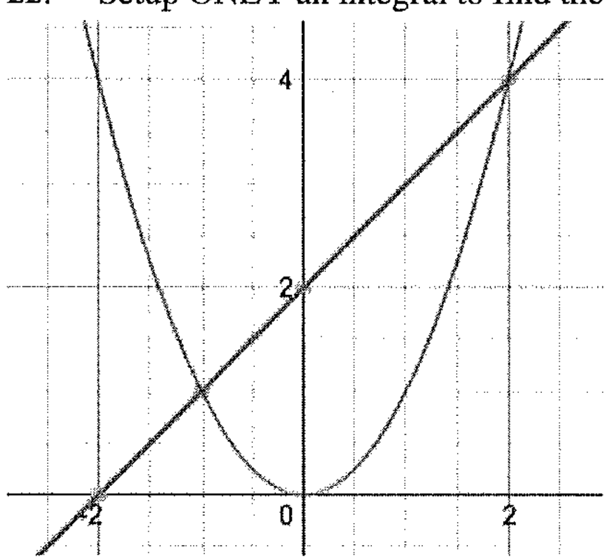
dimensions 45m X 30m

fencing 180m

21. A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose, and what are its dimensions? (page 222, #7)

Max Area = 80,000 m² dimensions 200m X 400m

22. Setup ONLY an integral to find the area between the curves $y = x^2$ and $y = x + 2$.



$\int_{-1}^{2} x+2-x^2\,dx$

Area using rectangles- class notes

Area of region between curves- class notes