Section 4.3 Definite Integrals and Area

Let f be any continuous function over $[a, b]$ and F be any antiderivative of f. Then the definite integral of f from a to b is $\int_a^b f(x)dx = F(b) - F(a)$.

Example 1: Evaluate the following.

a) $\int_{-1}^{4}(x^2 - x)\,dx$

b) $\int_0^2 e^x dx$

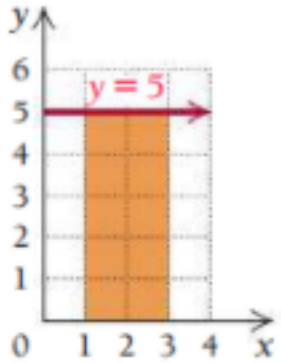
c) $\int_2^5 \frac{1}{x} dx$

d) $\int_{-4}^{-1} \frac{1}{x} dx$

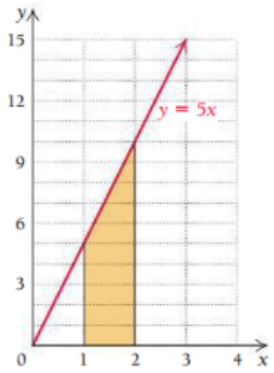
The area between the x-axis and the graph of the non-negative continuous function $y = f(x)$ over $[a, b]$ is found by evaluating the definite integral $\int_a^b f(x)dx = F(b) - F(a)$ where F is an antiderivative of f.

Example 2: Finding Area using Definite Integrals

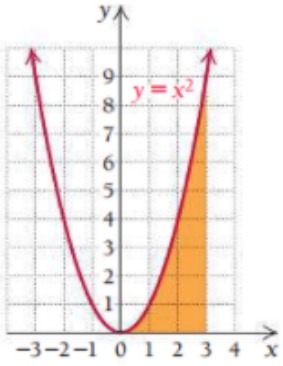
a) $y = 5;\ [1, 3]$



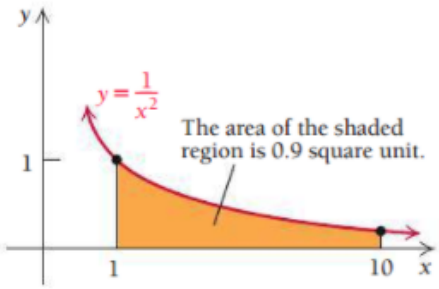
b) $y = 5x;\ [1, 2]$



c) $y = x^2;$ [0,3]



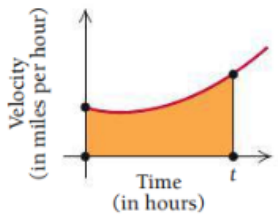
d) $y = \frac{1}{x^2}$ over [1, 10].



Section 4.3 Definite Integrals (Continued)

Example 3: Explain what the shaded area represents.

15.



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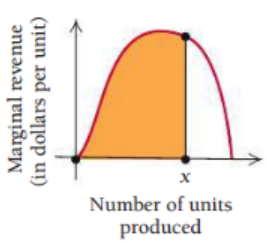

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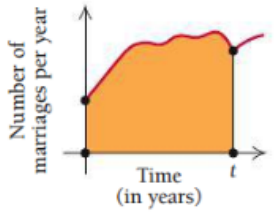

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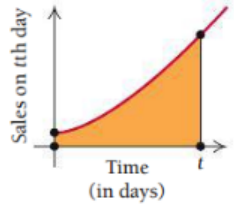
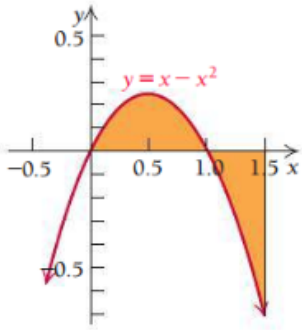


If a function has areas both below and above the x-axis, the definite integral gives the net total area, or the difference between the sum of the areas above the x-axis and the sum of the areas below the x-axis.

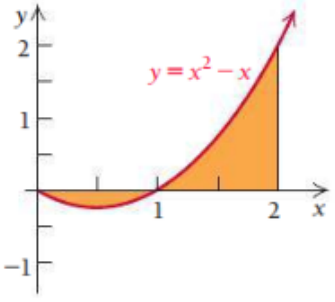
- If there is more area above the x-axis than below, then the definite integral is _____
- If there is more area below the x-axis than above, then the definite integral is _____
- If the areas above and below the x-axis are the same, then the definite integral is _____

Example 4: Evaluate each integral. Then state whether the result indicates that there is more area above or below the x-axis or that the areas above and below the axis are equal.

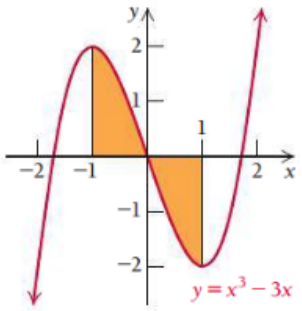
35. $\int_0^{1.5} (x - x^2)\,dx$



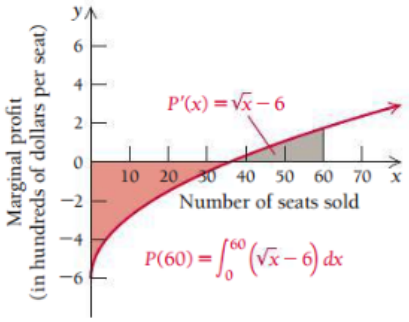
36. $\int_0^{2} (x^2 - x)\,dx$



37. $\int_{-1}^{1} (x^3 - 3x)\,dx$



Example 5: Northeast Airlines determines that the marginal profit, in hundreds of dollars per seat, from the sale of x seats on a jet traveling from Atlanta to Kansas City is given by $P'(x) = \sqrt{x} - 6$. Find the total profit when 60 seats are sold.



Example 6: Kitchens-to-Please Contracting determines that the marginal cost, in dollars per square foot, of installing x square feet of kitchen countertop is given by $C'(x) = 4\sqrt[3]{x}$.

a) Find the cost of installing 50 square feet of countertop.

b) Find the cost of installing an extra 14 square feet of countertop after 50 square feet have already been installed.

Example 7: Melanie's Crafts estimate that its sales are growing continuously at a rate given by $S'(t) = 20e^t$, where $S'(t)$ is in dollars per day, on day t.

a) Find the accumulated sales for the first 5 days

b) Find the accumulated sales from the beginning of the 2nd day through the 5th day.

4.3

Section ~~4.2~~ Definite Integrals and Area

Let f be any continuous function over $[a, b]$ and F be any antiderivative of f. Then the definite integral of f from a to b is $\int_a^b f(x)dx = F(b) - F(a)$.

* don't need to write +C with definite integrals

Example 1: Evaluate the following.

a) $\int_{-1}^{4}(x^2 - x)\,dx = \frac{1}{3}x^3 - \frac{1}{2}x^2 \Big|_{-1}^{4} = \left(\frac{1}{3}\cdot 4^3 - \frac{1}{2}\cdot 4^2\right) - \left(\frac{1}{3}\cdot(-1)^3 - \frac{1}{2}(-1)^2\right)$

means evaluate

$= \left(\frac{64}{3} - 8\right) - \left(-\frac{1}{3} - \frac{1}{2}\right) = \frac{40}{3} - \left(-\frac{5}{6}\right)$

$= \frac{85}{6}$

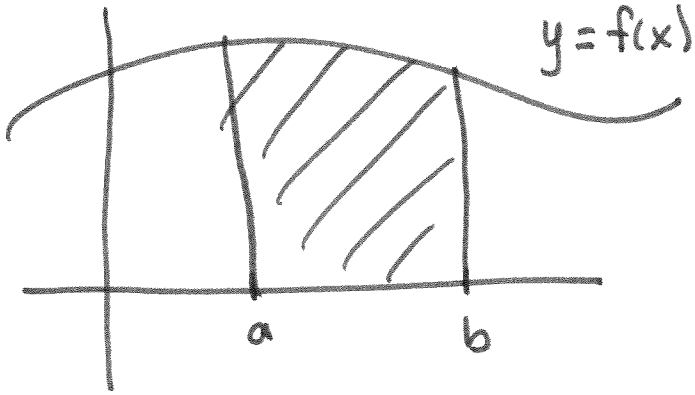
b) $\int_0^2 e^x dx = e^x \Big|_0^2 = e^2 - e^0 = 6.389$

c) $\int_2^5 \frac{1}{x}dx = \ln|x| \Big|_2^5 = \ln 5 - \ln 2 = .916$

d) $\int_{-4}^{-1} \frac{1}{x}dx = \ln|x| \Big|_{-4}^{-1} = \ln|-1| - \ln|-4| = \ln 1 - \ln 4$

$= -1.386$

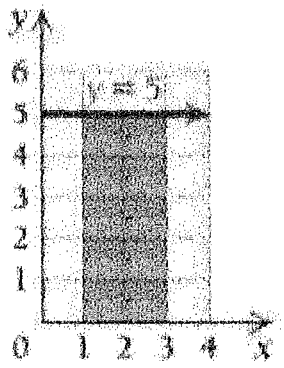
The area between the x-axis and the graph of the non-negative continuous function $y = f(x)$ over $[a, b]$ is found by evaluating the definite integral $\int_a^b f(x)dx = F(b) - F(a)$ where F is an antiderivative of f.



$$A = \int_a^b f(x)\,dx = F(b) - F(a)$$

Example 2: Finding Area using Definite Integrals

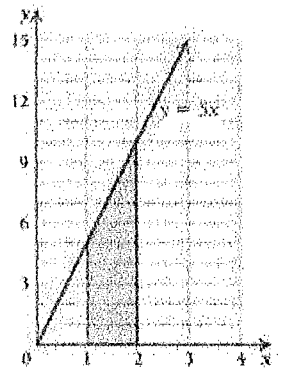
a) $y = 5;\ [1, 3]$



$$A = \int_1^3 5\,dx = 5x\Big|_1^3 = 5(3) - 5(1) = 10$$

constant rule

b) $y = 5x;\ [1, 2]$

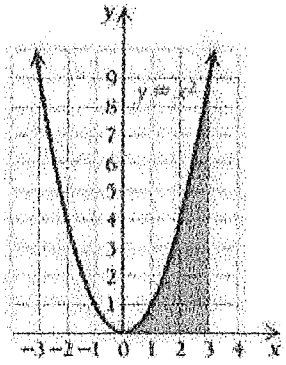


$$A = \int_1^2 5x\,dx = \frac{5}{2}x^2\Big|_1^2 = \frac{5}{2}\cdot 2^2 - \frac{5}{2}\cdot 1^2$$

power rule

$$= \frac{20}{2} - \frac{5}{2} = \frac{15}{2}$$

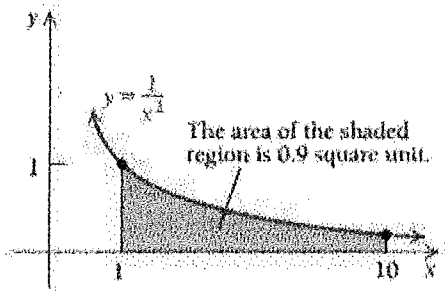
c) $y = x^2;\ [0,3]$



$$A = \int_0^3 x^2\,dx = \frac{1}{3}x^3\Big|_0^3 = \frac{1}{3}\cdot 3^3 - \frac{1}{3}\cdot 0^3$$

$$= \frac{27}{3} - 0 = 9$$

d) $y = \frac{1}{x^2}$ over [1, 10].



$$A = \int_1^{10} \frac{1}{x^2}\,dx = \int_1^{10} x^{-2}\,dx = -x^{-1}\Big|_1^{10}$$

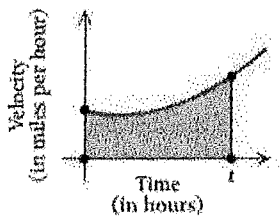
$$= \left(-10^{-1}\right) - \left(-1^{-1}\right) = \left(-\frac{1}{10}\right) - \left(\frac{-1}{1}\right)$$

$$= -\frac{1}{10} + 1 = \frac{9}{10} = .9$$

Section 4.3 Definite Integrals (Continued)

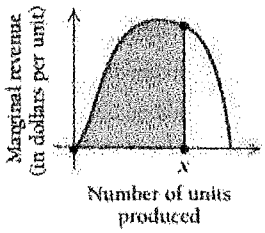
Example 3: Explain what the shaded area represents.

15.



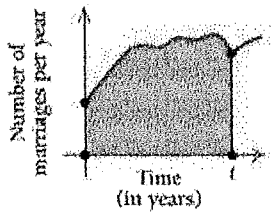
total number of miles traveled in t hours

19.



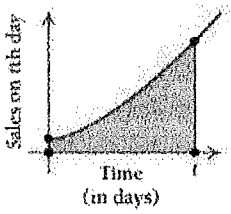
total revenue, in dollars, when x units are produced

17.



total number of marriages in t years

21.



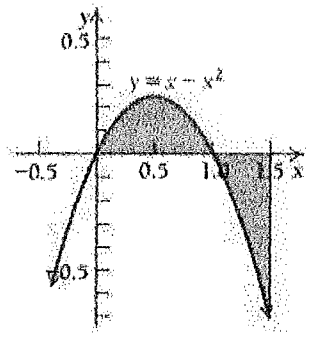
total sales in t days

If a function has areas both below and above the x-axis, the definite integral gives the net total area, or the difference between the sum of the areas above the x-axis and the sum of the areas below the x-axis.

- If there is more area above the x-axis than below, then the definite integral is positive
- If there is more area below the x-axis than above, then the definite integral is negative
- If the areas above and below the x-axis are the same, then the definite integral is zero

Example 4: Evaluate each integral. Then state whether the result indicates that there is more area above or below the x-axis or that the areas above and below the axis are equal.

35. $\int_0^{1.5} (x - x^2)\,dx$



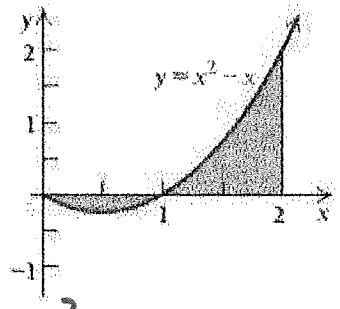
$\int_0^{1.5} x - x^2\,dx$

$\frac{1}{2}x^2 - \frac{1}{3}x^3 \Big|_0^{1.5}$

$\left(\frac{1}{2}(1.5)^2 - \frac{1}{3}(1.5)^3\right) - (0)$

0 above & below equal

36. $\int_0^2 (x^2 - x)\,dx$



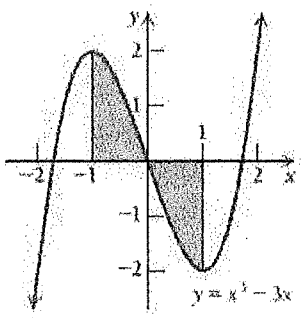
$\int_0^2 x^2 - x\,dx$

$\frac{1}{3}x^3 - \frac{1}{2}x^2 \Big|_0^2$

$\left(\frac{8}{3} - \frac{4}{2}\right) - (0 - 0)$

$\frac{2}{3}$ more above

37. $\int_{-1}^{1} (x^3 - 3x)\,dx$



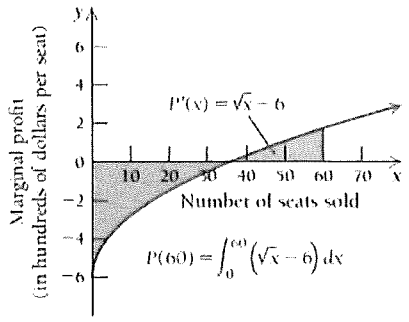
$\int_{-1}^{1} x^3 - 3x\,dx$

$\frac{1}{4}x^4 - \frac{3}{2}x^2 \Big|_{-1}^{1}$

$\left(\frac{1}{4}(1)^4 - \frac{3}{2}(1)^2\right) - \left(\frac{1}{4}(-1)^4 - \frac{3}{2}(1)^2\right)$

0 above & below equal

Example 5: Northeast Airlines determines that the marginal profit, in hundreds of dollars per seat, from the sale of x seats on a jet traveling from Atlanta to Kansas City is given by $P'(x) = \sqrt{x} - 6$. Find the total profit when 60 seats are sold.



$$\int_0^{60} \sqrt{x} - 6\,dx = \int_0^{60} x^{1/2} - 6\,dx$$

$$= \frac{2}{3}x^{3/2} - 6x \Big|_0^{60} = \left(\frac{2}{3}\cdot 60^{3/2} - 6\cdot 60\right) - (0-0)$$

$$= -50.1613$$

When 60 seats are sold, Profit is -5016.13 (loss)

Example 6: Kitchens-to-Please Contracting determines that the marginal cost, in dollars per square foot, of installing x square feet of kitchen countertop is given by $C'(x) = 4\sqrt[3]{x}. = 4x^{1/3}$

a) Find the cost of installing 50 square feet of countertop.

b) Find the cost of installing an extra 14 square feet of countertop after 50 square feet have already been installed.

a) $$\int_0^{50} 4x^{1/3}\,dx = 4\cdot\frac{3}{4}x^{4/3} = 3x^{4/3}\Big|_0^{50} = 3\cdot 50^{4/3} - 3\cdot 0^{4/3}$$
$$= \$552.60$$

b) $$\int_{50}^{64} 4x^{1/3}\,dx = 3x^{4/3}\Big|_{50}^{64} = 3\cdot 64^{4/3} - 3\cdot 50^{4/3} = \$215.40$$

Example 7: Melanie's Crafts estimate that its sales are growing continuously at a rate given by $S'(t) = 20e^t$, where $S'(t)$ is in dollars per day, on day t.

a) Find the accumulated sales for the first 5 days

b) Find the accumulated sales from the beginning of the 2nd day through the 5th day. ☆

a) $$\int_0^5 20e^t\,dt = 20e^t\Big|_0^5 = 20e^5 - 20e^0 = 2948.26$$

b) $$\int_1^5 20e^t\,dt = 20e^t\Big|_1^5 = 20e^5 - 20e^1 = \underline{2913.90}$$

day 5 — day 1 — days 2-5